Distribution Functions for Random Variables: Lecture VII

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REI
co-op

(a) Little Fish in Florida

(b) Artic Grayling

Reading Question

- From the discussion in the text, demonstrate that the distribution function

$$f(x,y) = \frac{3}{2}\left(x^2 + y^2\right) \text{ for } x, y \in [0,1] \tag{1}$$

is always positive

Bivariate Continuous Random Variables

- **Definition 3.4.1.** If there is a nonnegative function $f(x,y)$ defined over the whole plane such that

$$P\left(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2\right) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f\left(x,y\right) dx dy \tag{2}$$

  for x_1, x_2, y_1, and y_2 satisfying $x_1 \leq x_2$, $y_1 \leq y_2$, then (X,Y) is a bivariate continuous random variable and $f\left(X,Y\right)$ is called the joint density function.
- Much of the work with distribution functions involves integration. In order to demonstrate a couple of solution techniques, I will work through a couple of examples.
  - **Example 3.4.1.** If $f\left(x,y\right) = xy \exp\left(-x-y\right)$, $x > 0$, $y > 0$ and 0 otherwise, what is $P\left(X > 1, Y < 1\right)$

- First, note that the integral can be separated into two terms

$$P(X > 1, Y < 1) = \int_1^\infty xe^{-x}dx \int_0^1 ye^{-y}dy \tag{4}$$

- Each of these integrals can be solved using integration by parts:

$$\begin{aligned} d(uv) &= v\,du + u\,dv \\ v\,du &= d(uv) - u\,dv \\ \int v\,du &= uv - \int u\,dv \end{aligned} \tag{5}$$

- In terms of a proper integral we have

$$\int_a^b v\,du = \left(uv\right|_a^b - \int_a^b u\,dv \tag{6}$$

- In this case, we have

$$\int_1^\infty xe^{-x}dx \Rightarrow \begin{cases} v = x,\, dv = 1 \\ du = e^{-x},\, u = -e^{-x} \end{cases}$$

$$\int_1^\infty xe^{-x}dx = \left(-xe^{-x}\right|_1^\infty + \int_1^\infty e^{-x}dx = 2e^{-1} = 0.74 \tag{7}$$

- Working on the second part of the integral

$$\begin{aligned}\int_0^1 ye^{-y}dy &= \left(-ye^{-1}\right|_0^1 + \int_0^1 e^{-y}dy \\ &= \left(-ye^{-1}\right|_0^1 + \left(-e^{-y}\right|_0^1 \\ &= \left(-e^{-1}+0\right) + \left(-e^{-1}+1\right)\end{aligned} \tag{8}$$

- Putting the two parts together

$$\begin{aligned}P\left(X>1, Y<1\right) &= \int_1^\infty xe^{-x}dx \int_0^1 ye^{-y}dy \\ &= (0.735)(0.264) = 0.194\end{aligned} \tag{9}$$

- **Example 3.4.3.** This example demonstrates the use of changes in variables. Implicitly the example assumes that the random variables are joint uniform for all $0 < x$, $y < 1$. The question is then: What is the probability that $x^2 + y^2 < 1$? Mathematically, this question is not separable:

$$P\left(X^2 + Y^2 < 1\right) = \int_0^1 \left(\int_0^{\sqrt{1-x^2}} dy\right) dx = \int_0^1 \sqrt{1 - x^2} dx \quad (10)$$

- As previously stated, we will solve is problem using integration by change in variables. By trigonometric identity $1 = \sin^2 x + \cos^2 x$. Therefore, $\sin^2 x = 1 - \cos^2 x$ or $\sin(x) = \sqrt{1 - \cos^2(x)}$. The change in variables is then to let $\phi(x) = \cos(x)$. The integration by change in variables is then:

$$\int_{x_1}^{x_2} f(x)\, dx = \int_{t_1}^{t_2} f[\phi(t)]\, \phi'(t)\, dt \tag{11}$$

Think of the chain rule for the derivative

$$dF(\phi(x)) = F'(\phi(x))\, \phi'(x) \tag{12}$$

Numeric Values of the Transformation

$$\phi\left(x\right) = \cos\left(x\right) \Rightarrow \sqrt{1-x^2} \approx \sqrt{1-\cos\left(z\right)^2} \tag{13}$$

x	$z = x\pi/2$	$\sqrt{1-x^2}$	$\sin\left(z\right)\sqrt{1-\cos\left(z\right)^2}$
0.1000	0.1571	0.9950	0.0245
0.2000	0.3142	0.9798	0.0955
0.3000	0.4712	0.9539	0.2061
0.4000	0.6283	0.9165	0.3455
0.5000	0.7854	0.8660	0.5000
0.6000	0.9425	0.8000	0.6545
0.7000	1.0996	0.7141	0.7939
0.8000	1.2566	0.6000	0.9045
0.9000	1.4137	0.4359	0.9755
1.0000	1.5708	0.0000	1.0000

- First, we let $\phi(x) = \cos(x)$. Hence,

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}dx &= \int_{\phi(t_1)}^{\phi(t_2)} \sin(\theta)\sqrt{1-\cos^2(\theta)}\,d\theta \\ &= \int_{\phi(t_1)}^{\phi(t_2)} \sin^2(\theta)\ d\theta\end{aligned} \tag{14}$$

- Such that $\phi(t_1) = x_1$ and $\phi(t_2) = x_2$

$$\phi^{-1}(0) = \frac{\pi}{2}\,\text{and}\,\phi^{-1}(1) = 0 \tag{15}$$

- The problem becomes

$$\int_0^1 \sin^2(\theta)\,d\theta = \left[-\frac{1}{2}\sin(\theta)\cos(\theta) + \frac{1}{2}\right|_0^{\frac{\pi}{2}} = \frac{\pi}{4} \tag{16}$$

Marginal Density

- **Theorem 3.4.1.** Let $f(x,y)$ be the joint density of X and Y and let $f(x)$ be the marginal density of X. Then

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \tag{17}$$

- Going back to the distribution function from example 3.4.1, we have:

$$f(x,y) = xye^{-(x+y)} \tag{18}$$

- Proving that this is a proper distribution function, if we limit our consideration to only nonnegative x and y , $f(x,y)$ is always positive. From our discussion last time, it is also obvious that

$$\int_0^\infty \int_0^\infty f(x,y)\,dxdy = \left(\int_0^\infty xe^{-x}dx\right)\left(\int_0^\infty ye^{-y}dy\right)$$

$$= \left(-\left(xe^{-x}\right|_0^\infty + \int_0^\infty e^{-x}dx\right)\left(-\left(ye^{-y}\right|_0^\infty + \int_0^\infty e^{-y}dy\right)$$

$$= \left(-\left(\infty\cdot 0 - 0\cdot 1\right) - (0-1)\right)\left(-\left(\infty\cdot 0 - 0\cdot 1\right) - (0-1)\right) \tag{19}$$

- Thus, this is a proper density function. The marginal density function for x follows this formulation:

$$f(x) = \int_0^\infty f(x,y)\,dy = \left(xe^{-x}\right)\int_0^\infty ye^{-y}dy$$

$$= \left(xe^{-x}\right)\left(-\left(ye^{-y}\right|_0^\infty + \int_0^\infty e^{-y}dy\right) \quad (20)$$

$$= xe^{-y}$$

- For example 3.4.5,

$$f(x,y) = \frac{3}{2}\left(x^2+y^2\right) \quad (21)$$

- To derive the marginal density for x:

$$\begin{aligned} f(x) &= \int_0^1 \frac{3}{2}\left(x^2+y^2\right)dy \\ &= \frac{3}{2}x^2\int_0^1 dy + \tfrac{3}{2}\int_0^1 y^2 dy \\ &= \frac{3}{2}x^2\left(y\right|_0^1 + \tfrac{3}{2}\left(\tfrac{1}{3}y^3\right|_0^1 \\ &= \frac{3}{2}\left(x^2+\tfrac{1}{3}\right) \end{aligned} \tag{22}$$

- **Definition 3.4.3.** Let (X, Y) have the joint density $f(x, y)$ and let S be a subset of the plane which has a shape as in Figure 3.3. We assume that $P[(X, Y) \in S] > 0$. Then the conditional density of X given $(X, Y) \in S$, denoted $f(x|S)$, is defined by

$$f(x|S) = \begin{cases} \dfrac{\displaystyle\int_{h(x)}^{g(x)} f(x, y)\, dy}{P[(X, Y) \in S]} \text{ for } a \leq x \leq b, \\ \\ 0 \text{ otherwise} \end{cases} \tag{23}$$

- **Example 3.4.8.** Suppose $f(x,y) = 1$ for $0 \leq x \leq 1$, $0 \leq y \leq 1$, and 0 otherwise. Obtain $f(x|X < Y)$.

$$
\begin{aligned}
f(x|X<Y) &= \frac{\displaystyle\int_x^1 dy}{\displaystyle\int_0^1 \int_x^1 dy\,dx} \\
&= \frac{\left(y\right|_x^1}{\displaystyle\int_0^1 \left(y\right|_x^1 dx} \\
&= \frac{1-x}{\displaystyle\int_0^1 (1-x)\,dx} \\
&= \frac{1-x}{\left(x - \frac{1}{2}x^2\right|_0^1} \\
&= 2(1-x)
\end{aligned}
\tag{24}
$$

- Amemiya Conditional Density [pp. 35-38], Moss [pp.67-70]
- **Definition 3.4.4.** The conditional probability that X falls into $[x_1, x_2]$ given $Y = y_1 + cX$ is defined by

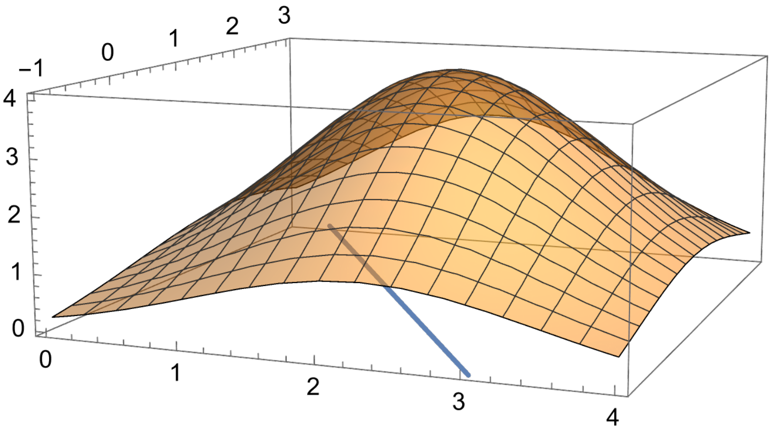
$$P\left(x_1 \leq X \leq x_2 | Y = y_1 + cX\right) = \lim_{y_2 \rightarrow y_1} P\left(x_1 \leq X \leq x_2 | y_1 + cX \leq Y \leq y_2 + cX\right) \tag{25}$$

for all x_1, x_2 satisfying $x_1 \leq x_2$.

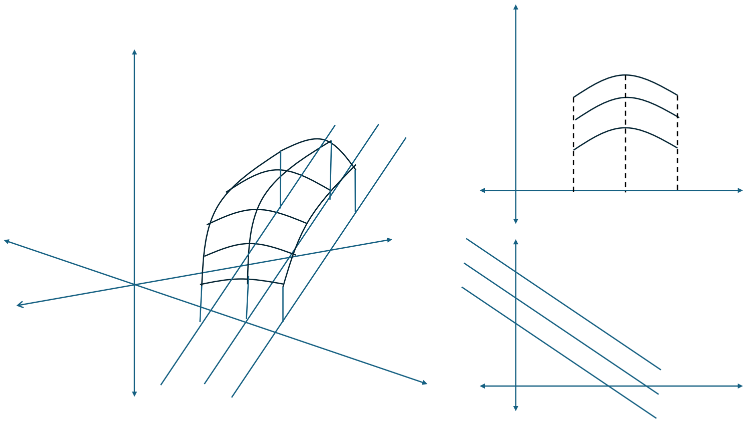
- **Definition 3.4.5.** The conditional density of X given $Y = y_1 + cX$, denoted by $f\left(x|Y = y_1 + cX\right)$, if it exists, is defined to be a function that satisfies

$$P\left(x_1 \leq X \leq x_2 | Y = y_1 + cX\right) = \int_{x_1}^{x_2} f\left(x|Y = y_1 + cX\right) dx \quad (26)$$

Multidimensional Distribution Function with a Line



A Region around the Line



- **Theorem 3.4.2.** The conditional density $f\left(x|Y=y_1+cX\right)$ exists and is given by

$$f\left(x|Y=y_1+cX\right)=\frac{f\left(x,y_1+cx\right)}{\int_{-\infty}^{\infty}f\left(x,y+cx\right)dx} \tag{27}$$

provided the denominator is positive.

  - Proof: We have

$$\lim_{y_2\to y_1}P\left(x_1\leq X\leq x_2|y_1+cX\leq Y\leq y_2+cX\right)$$

$$=\lim_{y_2\to y_1}\frac{\int_{x_1}^{x_2}\int_{y_1+cx}^{y_2+cx}f\left(x,y\right)dydx}{\int_{-\infty}^{\infty}\int_{y_1+cx}^{y_2+cx}f\left(x,y\right)dxdy} \tag{28}$$

- **Theorem 3.4.2.** Proof continued
  - Mean value theory of integration: Let f be a continuous function on the closed interval $[a,b]$. Then there is some number X such that $a \leq X \leq b$ such that

$$\int_a^b f(x)\,dx = (b-a)\,f(X) \tag{29}$$

  - Thus, by the mean value of integration

$$\lim_{y_2 \to y_1} \frac{\int_{x_1}^{x_2} \int_{y_1+cx}^{y_2+cx} f(x,y)\,dydx}{\int_{-\infty}^{\infty} \int_{y_1+cx}^{y_2+cx} f(x,y)\,dxdy} = \lim_{y_1 \to y_2} \frac{\int_{x_1}^{x_2} f(x, y^* + cx)\,dx}{\int_{-\infty}^{\infty} f(x, y^* + cx)\,dx} \tag{30}$$

  where $y_1 \leq y^* \leq y_2$. As $y_2 \to y_1$, $y^* \to y_1$.

- Hence

$$\lim_{y_1 \to y_2} \frac{\displaystyle\int_{x_1}^{x_2} f\left(x, y^* + cx\right) dx}{\displaystyle\int_{-\infty}^{\infty} f\left(x, y^* + cx\right) dx} = \frac{\displaystyle\int_{x_1}^{x_2} f\left(x, y_1 + cx\right) dx}{\displaystyle\int_{-\infty}^{\infty} f\left(x, y_1 + cx\right) dx} \tag{31}$$

- **Theorem 3.4.3.** The conditional density of X given $Y = y_1$, denoted by $f(x|y_1)$, is given by

$$f(x|y_1) = \frac{f(x, y_1)}{f(y_1)} \tag{32}$$

Independence

- **Definition 3.4.6.** Continuous random variable X and Y are said to be independent if $f(x,y) = f(x) f(y)$ for all x and y.
- **Theorem 3.4.4.** Let S be a subset of the plane such that $f(x,y) > 0$ over S and $f(x,y) = 0$ outside of S. Then X and Y are independent if and only if S is a rectangle (allowing $-\infty$ or ∞ to be an end point) with sides parallel to the axes and $f(x,y) = g(x)/h(y)$ over S, where $g(x)$ and $h(y)$ are some functions of x and y, respectively. Note that $g(x) = cf(x)$ for some c, $h(y) = c^{-1} f(y)$.
- **Definition 3.4.7.** A finite set of continuous random variables $X, Y, Z, \cdots$ are said to be mutually independent if

$$f(x,y,z,\cdot) = f(x) f(y) g(z) \cdots \tag{33}$$

Distribution Function

- **Definition 3.5.1.** The cumulative distribution function of a random variable X, denoted $F(x)$, is defined by

$$F(x) = P(X < x) \tag{34}$$

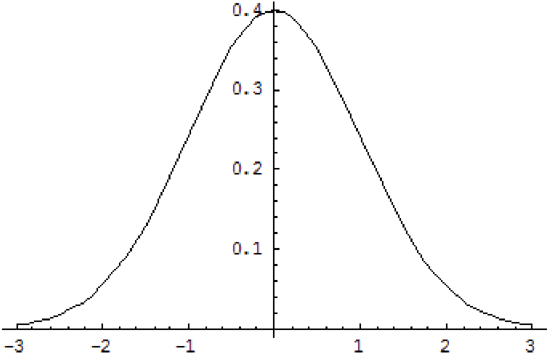
for every real x.

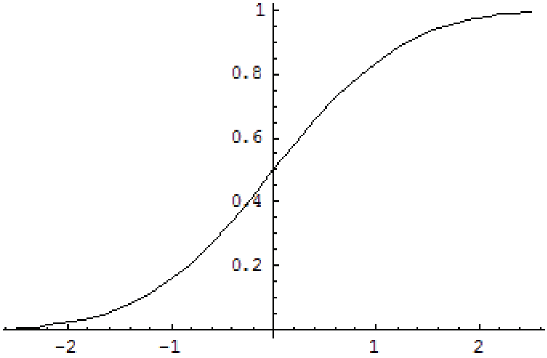
  - In the case of a continuous random variable:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt \tag{35}$$

  - For the uniform distribution:

$$F(x) = \int_{-\infty}^{x} dt \Rightarrow \begin{cases} 0 \text{ if } x \leq 0 \\ (t|_0^x \text{ if } 0 < x < 1 \\ 1 \text{ if } x > 1 \end{cases} \tag{36}$$


0.4
0.3
0.2
0.1
-3
-2
-1
1
2
3


1
0.8
0.6
0.4
0.2
-2
-1
1
2

Change of Variables

- Change in variables is a technique used to derive the distribution function for one random variable by transforming the distribution function of another random variable.
- **Theorem 3.6.1.** Let $f(x)$ be the density of X and let $Y = \phi(X)$, where ϕ is a monotonic differentiable function. Then the density $g(y)$ of Y is given by

$$g(y) = f\left[\phi^{-1}(y)\right] \cdot \left|\frac{d\,\phi^{-1}(y)}{d\,y}\right| \tag{37}$$

- **Example 3.6.1.** Suppose $f(x) = 1$ for $0 < x < 1$ and 0 otherwise. Assuming $Y = X^2$, what is the distribution function $g(y)$ for Y? First, it is possible to show that $Y = X^2$ is a monotonic or one-to-one mapping over the relevant range. Given this one-to-one mapping, it is possible to derive the inverse function:

$$\phi(x) = x^2 \Rightarrow \phi^{-1}(y) = \sqrt{y} \tag{38}$$

Following the definition:

$$g(y) = 1 \left| \frac{1}{2} y^{-1} 2 \right| = \frac{1}{2\sqrt{y}} \tag{39}$$

- **Theorem 3.6.2.** Suppose the inverse of $\phi(x)$ is multivalued and can be written as

$$x_i = \psi_i(y) \;\; i = 1, 2, ...n_y \tag{40}$$

Note that n_y indicates the possibility that the number of values of x varies with y. Then the density $g(y)$ of Y is given by

$$g(y) = \sum_{i=1}^{n_y} \frac{f[\psi_i(y)]}{|\phi'[\psi_i(y)]|} \tag{41}$$

where $f(.)$ is the density of X and ϕ' is the derivative of ϕ.

- **Theorem 3.6.3.** Let $f(x_1, x_2)$ be the joint density of a bivariate random variable (X_1, X_2) and let (Y_1, Y_2) be defined by a linear transformation

$$\begin{aligned} Y_1 &= a_{11}X_1 + a_{12}X_2 \\ Y_2 &= a_{21}X_1 + a_{22}X_2 \end{aligned} \tag{42}$$

Suppose $a_{11}a_{22} - a_{12}a_{21} \neq 0$ so that the equations can be solved for X_1 and X_2 as

$$\begin{aligned} X_1 &= b_{11}Y_1 + b_{12}Y_2 \\ X_2 &= b_{21}Y_1 + b_{22}Y_2 \end{aligned} \tag{43}$$

- Then the joint density $g(y_1, y_2)$ of (Y_1, Y_2) is given by

$$g(y_1, y_2) = \frac{f(b_{11}y_1 + b_{12}y_2, b_{21}y_1 + b_{22}y_2)}{|a_{11}a_{22} - a_{12}a_{21}|} \tag{44}$$

where the support of g, that is, the range of (y_1, y_2) over which g is positive must be appropriately determine

Readings

- Amemiya 3.5 – 3.7, Moss 3.4 – 3.7
- Popper - Chapter 3: Theories