Integrals and the Lebesgue Measure: Lecture IV

Charles B. Moss

August 28, 2026

Fish Wheel

My Catch

Explaining the Homework

- One of the things that we are interested in is developing a test to determine whether two distributions are different. Specifically, if I have $x \sim F$ and $z \sim G$, I may wish to test whether the distribution for x is the same as the distribution for z).
- You have actually done this, maybe not at the same level of rigor. Specifically, what if I think (believe or conjecture) that $x \sim \mathrm{N}\left(\mu_x, \sigma_x\right)$ and $z \sim \mathrm{N}\left(\mu_z, \sigma_z\right)$. For the moment, we will assume that $\sigma_x = \sigma_z = \sigma$. In this case testing for whether the distributions are the same involves the hypothesis that $\mu_x = \mu_z$.

Statistical Information Theory

- One approach to thinking about tests involves statistical information theory (Kullback) where we define the difference in the information between two distribution functions as

$$I\left(f,g\right)=\int_{-\infty}^{\infty}f\left(x\right)\ln\left(\frac{f\left(x\right)}{g\left(x\right)}\right)dx \tag{1}$$

  The concept is that if the two distribution are the same (or close enough) $I\rightarrow 0$.

- In many cases, this expression has a limiting distribution $I\sim\chi_k^2$.
- Kullback, Solomon. 1968. *Information Theory and Statistics* New York: Dover Publications.

Effect of Climate on Crop Yields

- Why this homework?
- Miao, Ruiqing, Madhu Khanna, and Hzixiao Huang. 2016. Reponsiveness of Crop Yield and Acreage to Prices and Climate. *American Journal of Agricultural Economics* 98(1): 191-211. [DOI: 10.1093/ajae/aav025]

More on Measures

- Note on the supremum and infemum – *De Morgan's Law*

$$\left(\bigcup_{n=0}^{\infty} A_n\right)^c = \bigcap_{n=0}^{\infty} A_n^C \tag{2}$$

In addition, we define

$$\lim_{n\to\infty} \sup A_n = \bigcap_{n=0}^{\infty} \bigcup_{m=n}^{\infty} A_m \quad \lim_{n\to\infty} \inf A_n = \bigcup_{n=0}^{\infty} \bigcap_{m=n}^{\infty} A_m \tag{3}$$

- **Proposition 1.6 (Continuity and Nondecreasing Sequences)** If μ is additive on a ring $\mathfrak{A}$, then (i) $\Leftrightarrow$ (ii), where:
  - i μ is σ-additive;
  - ii $(A_n) \subset \mathfrak{A}$ and $A \in \mathfrak{A}$, $A_n \uparrow A \Rightarrow \mu(A_n) \uparrow \mu(A)$.

- **Proposition 1.7 (Continuity on Nonincreasing Sequences)** Let μ be σ-additive on a ring $\mathfrak{A}$. Then

$$(A_n) \subset \mathfrak{A} \text{ and } A \in \mathfrak{A}, A_n \downarrow A, \mu(A_n) < \infty \Rightarrow \mu(A_n) \downarrow \mu(a) \quad (4)$$

- **Corollary 1.8 (Upper and Lower Semicontinuity of the Measure)** Let μ be σ-additive on a σ-algebra $\mathfrak{E}$ and let $(A_n) \subset \mathfrak{E}$. Then we have

$$\mu\left(\lim_{n\to\infty} \inf A_n\right) \leq \lim_{n\to\infty} \inf \mu(A_n) \quad (5)$$

and, if $\mu(X) < \infty$, we have also

$$\lim_{n\to\infty} \sup \mu(A_n) \leq \mu\left(\lim_{n\to\infty} \sup A_n\right) \quad (6)$$

Measurable Spaces and Measure Spaces

- Let $\mathfrak{E}$ be a σ-algebra of subsets on X. Then we say that the pair $(x, \mathfrak{E})$ is a *measurable space*.
- Let $\mu : \mathfrak{E} \rightarrow [0, +\infty]$ be a σ-additive function. Then we call μ a *measure* on $(X, \mathfrak{E})$, and we call the triple $(X, \mathfrak{E}, \mu)$ a *measure space*.
- The measure μ is said to be *finite* if $\mu(X) < \infty$, σ-finite if there exists a sequence $(A_n) \subset \mathfrak{E}$ such that $\cup_n A_n = X$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$.
- μ is called the *probability measure* if $\mu(X) = 1$.
- **Dirac Mass** δ as a measure

$$\delta_x(B) = \begin{cases} 1 \text{ if } x \in B \\ 0 \text{ if } x \notin B \end{cases} \tag{7}$$

The probability number of a non-random variable.

The Fundamental Theorem of Calculus

- As a starting point consider the statement that

$$F\left(x\right)=\int_{a}^{b}f\left(t\right)dt \tag{8}$$

where $f\left(t\right)=F'\left(x\right)$ is defined as the derivative of $F\left(x\right)$.

- Consider how this definition is constructed. We start by defining the derivative of the function as

$$F'\left(x_i\right)=\lim_{\Delta x_i\rightarrow 0}\frac{F\left(x_i+\Delta x_i\right)-F\left(x_i\right)}{\Delta x_i} \tag{9}$$

- Next, we chop the integral up into parts starting with $x_i = x_0$

$$\begin{array}{c} F(x) = [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \\ \cdots [F(x_1) - F(x_0)] = F(x_n) - F(x_0), \; x = (a,b) \end{array} . \tag{10}$$

  Setting $x_n = b$ and $x_0 = a$.

- The clinch of the proof is to substitute the derivative for each increment

$$\begin{aligned} F(x_n + \Delta x_n) - F(x_n) &= \left[\frac{F(x_n + \Delta x_n) - F(x_n)}{\Delta x_n} \right] \Delta x_n \\ &= F'(x_n) \Delta x_n \end{aligned} \tag{11}$$

- The overall sum can then be written as

$$F(b) - F(a) = \sum_{i \ni : \max \Delta x_i \to 0} F'(x_i) \Delta x_i \tag{12}$$

- Our typical approach to integration is the relationship that this sum is defined for $F(x)$ such that $f(x) = F'(x)$.
- That is $f(x)$ is the derivative of $F(x)$. $F(x)$ is the anti-derivative of $f(x)$.

Linkage to the Riemann Sum

- Using the forgoing discussion, we can link the integral to the Riemann Sum.
- Consider two specifications

$$\begin{aligned} S_1(K) &= \sum_k |x_k - x_{k-1}| \sup_{[x_{k-1},x_k]} f(x) \\ S_2(k) &= \sum_k |x_k - x_{k-1}| \inf_{[x_{k-1},x_k]} f(x) \end{aligned} \tag{13}$$

The integral or the Riemann sum is defined as the quantity where $S_1(K) \rightarrow^- S_2(K)$ [Moss 46].

- Looking forward [Moss 49] we can create a Lebesgue measure by chopping up the original set into 'lengths.' For example, we start by considering the set $X \in [0,1]$. We consider subsets of this range $A_n = (a_n, b_n)$. The Lebesgue measure is then defined by the length

$$\begin{aligned} \lambda(B) &= \inf_{B \supset \cup_{j=1}^{\infty}(a_i, b_j)} \sum_{j=1}^{\infty} \lambda((a_j, b_j)) \\ &= \inf_{B \subset \cup_{j=1}^{\infty}(a_j, b_j)} \sum_{j=1}^{\infty} (b_j - a_j) \end{aligned} \tag{14}$$

Hence, the probability can be defined using the Lebesgue measure as

$$\int_A f(x)\,dx = \sup \sum_{m=1}^{n} \left(\inf_{x \in B_n} f(x) \right) \lambda(B_n) \tag{15}$$

Axiomatic Foundations

- A set $\{\omega_{j_1},\ldots\omega_{j_k}\}$ of different combinations of outcomes is called an event. These events could be simple events or compound events. In the Texas lotto case, the important aspect is that the event is something you could bet on (for example, you could bet on three numbers in the draw 35,20,15).
- A collection of events F is called a family of subsets of sample space Ω. This family consists of all possible subsets of Ω including Ω itself and the null-set $\emptyset$.
  - Following the betting line, you could bet on all possible numbers (covering the board) so that Ω is a valid bet.
  - Alternatively, you could bet on nothing, or $\emptyset$ is a valid bet.

Axiomatic Foundations - Continued

- Next, we will examine a variety of closure conditions. These are conditions that guarantee that if one set is an contained in a family, another related set must also be contained in that family.
  - First, we note that the family is closed under complimentarity

$$\text{If } A \in F \text{ then } \tilde{A} = \Omega \backslash A \in F \tag{16}$$

In this case $\tilde{A} = \Omega \backslash A \in F$ denotes all elements of Ω that are not contained in A.

- Second, we note that the family is closed under union

$$\text{If } A, B \in F \text{ then } A \cup B \in F \tag{17}$$

- Definition 1.1 (Bierens): A collection of subsets of a nonempty set satisfying closure under complementarity and closure under union is called an algebra.
- Adding closure under infinite union defined as

$$\text{If } A_j \in F \text{ for } j = 1, 2, 3, \ldots \text{ then } \bigcup_{j=1}^{\infty} A_j \in F \tag{18}$$

- Definition 1.2 (Bierens): A collection F of subsets of a nonempty set Ω satisfying closure under complementarity and infinite union is called a σ-algebra (sigma-algebra) or a Borel Field.

- We typically think of this as an odds function (i.e., what are the odds of a winning lotto ticket? 1/15,890,700).
- To be mathematically precise, suppose we define a set of events $(A = \{\ \omega_1, \ldots, \omega_j\}\ \in \Omega)$, say that we choose n different numbers. The probability of winning the lotto is $p[A] = P[\{\ \omega_1, \ldots, \omega_n\}\ ] = n/N$.
    - Our intuition would indicate that $P[\Omega] = 1$, or the probability of winning given that you have covered the board is equal to one (a certainty).
    - Further, if you don't bet the probability of winning is zeros or $P[\emptyset] = 0$.

- Definition 1.2.2: Given a sample space Ω and an associated σ-algebra F , a *probability function* is a function $P[A]$ with domain F that satisfies
  - $P[A] \geq 0$ for all $A \in F$.
  - $P[\Omega] = 1$
  - If $A_1, A_2, \cdots \in F$ are pairwise disjoint (i.e., $A_i \cap A_j = \emptyset$ for all $i \neq j$)

$$P\left[\bigcup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} P[A_i] \tag{19}$$

Axioms of Probability

- $P[A] \geq 0$ for any event A.
- $P[S] = 1$ where S is the sample space.
- If $\{\ A\}\ \ i = 1, 2, \ldots$ are mutually exclusive (that its $A_i \cap A_j = \emptyset$ for all $i \neq j$, then $P[A_1 \cup A_2 \cup \ldots] = P[A_1] + P[A_2] + \ldots$.

- Definition 1.1.1: The set S of all possible outcomes of a particular experiment is called the *sample space* for the experiment.
- Definition 1.1.2: An *event* is any collection of possible outcomes of an experiment, that is, any subset of S (including S itself).
- Defining the subset relationship
  - $A \subset B \Leftrightarrow x \in A \Rightarrow x \in B$.
  - $A = B \Leftrightarrow A \subset B \,\mathrm{and}\, B \subset A$.
  - *Union*: The union of A and B, written $A \cup B$, the set of elements that belong either to A or B

$$A \cap B = \{\, x : x \in A \,\mathrm{or}\, x \in B\} \tag{20}$$

- *Intersection*: The intersection of A and B, written $A \cap B$, is the set of elements that belong to both A and B.

$$A \cap B = \{\ x : x \in B \,\text{and}\, x \in B\} \tag{21}$$

- *Complementation*: The complement of A, written A^C, is the set of all elements that are not in A.

$$A^C \in \{\ x : x \notin A\} \tag{22}$$

- Theorem 1.1.1: For any three events A, B, and C defined on a sample space S,
  - Commutativity: $A \cup B = B \cup A$, $A \cap B = B \cap A$.
  - Associativity: $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cup C) = (A \cup B) \cup C$.
  - Distrubative Laws: $A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$, $A \cup (B \cap C) = (A \cap B) \cup (A \cap C)$.
  - DeMorgan's Laws: $(A \cup B)^C = A^C \cup B^C, (A \cap B)^C = A^C \cup B^C$.