Down the Rabbit Hole – An Odessy into Advanced Mathematics: Lecture III

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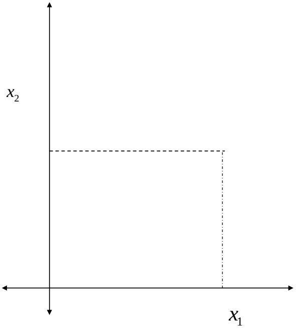
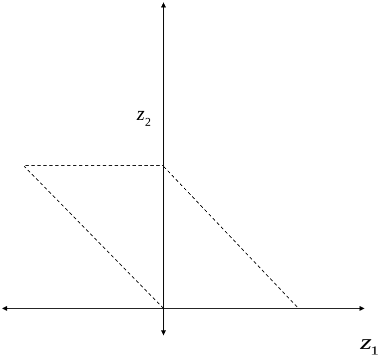
Gold Panning

Dredge 8

Starting with Two Simple Surfaces/Sets





Overview

- Reimann Sum

$$\begin{gathered}
\mathrm{P}\left[\left(x_0, x_1\right)\right] = \int_{x_0}^{x_1} f\left(x\right) dx \\
\approx \sum_{\left|\left|x_i\right|\right| \to 0} \inf\left(f\left(x_i\right), f\left(x_{i-1}\right)\right)\left|x_i - x_{i-1}\right| \\
A = \left\{\ \left(x_0, x_1\right), \left(x_1, x_2\right), \left(x_2, x_3\right)\right\} \ \Rightarrow \\
\left\{\ \mathrm{P}\left[A_1\right], \mathrm{P}\left[A_2\right], \mathrm{P}\left[A_3\right]\right\}
\end{gathered} \tag{1}$$

Mapping

- Note that the two sets are related

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_1 - x_2 \end{aligned} \tag{2}$$

- In matrix form

$$\begin{aligned} f &: x \rightarrow z \\ z &= \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} x \end{aligned} \tag{3}$$

Mapping – Changing Dimensions

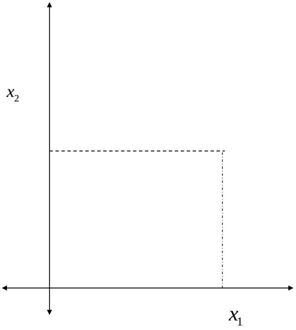
- Note that we can define mappings that change from one dimension to another.
- For example, we can map from the original x space which is $\Re^2$ to an $\Re^3$ space using the additional mapping function

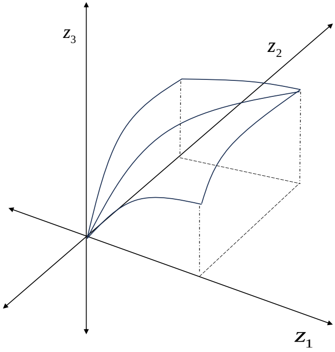
$$f_3(x) = x_1^2 + x_2^2 \tag{4}$$

The complete map would then be

$$f : x \to z \Rightarrow \begin{cases} f_1(x) = x_1 \\ f_2(x) = x_2 \\ f_3(x) = x_1^2 + x_2^2 \end{cases} \tag{5}$$

Graphics of Changing Dimensions





Defining a Metric Space

- A **Metric Space** is a combination of a set (or a space) S and a measure of distance (typically a norm).
- In the forgoing example, we can define a Metric space as the set of real numbers $\{ x_1, x_2 \} \in [0,1]$.
- We can then define the distance function as the general p norm

$$\|x\|_p = \left(\sum_{i=1}^{k} |x_i^p| \right)^{1/p} \tag{6}$$

- The Metric Space allows us to define an **Open ball**

$$A = \{ x : \|x - a\|_p < \epsilon \} \tag{7}$$

A Sequence of Open Balls

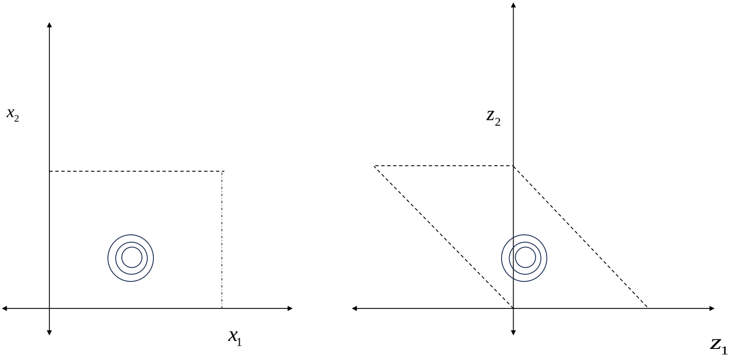
- Next, we define a sequence of open balls A_n where $n \in \mathbb{N}$ ($\mathbb{N}$ is the set of natural numbers).

$$(A_n) = \{\ x : \|a - a\|_p < \epsilon_n\} \tag{8}$$

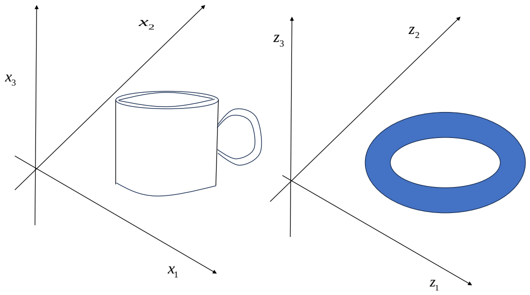
were $\epsilon_i > \epsilon_j$ when $j > i$.

- Given this sequence, the open ball converge $(A_n) \rightarrow x^*$.
- Taken together, the concept of Open Balls and covergence imply continuity.
  - If f in Equation 2 implies that A_n exists then B_n defined as $B_n = \{\ z : \|z - b\|_P < \epsilon^*\}$ exists and both have a point of convergence, then the map between A and B is continuous.

Continuous Mapping



Topology of the Torus



- Homeomorphism or isomorphism is a bijective and continuous function between two topological spaces (Wikipedia)

Functions of Sets

- As a starting point, consider sets defined on $A \in [0,1]$.
- For example, let $A = (a,b)$ where $a < b \, \text{and} \, a,b \in [0,1]$ $(\mu : A \rightarrow \Re)$.
- Based on these concepts, we can define

$$\mu\left(A\right) = b - a \tag{9}$$

  Note that $\mu\left(A\right) \geq 0$ with equality as $b \rightarrow a$.
- With a little more rigor $\mu : [0,1] \rightarrow [0,1]$ is a continuous function defined on $x \in [0,1]$.
- A little more complicated example $x = \{\, x_1, x_2 \}$ we can define

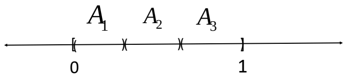
$$\begin{gathered} \mu\left(x,y\right) = \left(x_2 - y_2\right) \times \left(x_1 - y_1\right) \\ \text{where } x_2 \geq y_2 \text{ and } x_1 \geq y_1 \end{gathered} \tag{10}$$

Numerics

- For the first function, it is obvious that if $A = \{\ 0.50, 0.75\}$ then $\mu(A) = 0.25$.
- For the second function, note that $\mu : \Re^2 \rightarrow \Re^1$. Defining the subspace where $y = \{\ 0.5, 0.5\}$ and $x = \{\ 0.75, 0.75\}$, $\mu(Z) = 0.0625$ where $Z = \{\ 0.5 \leq x_1 \leq 0.75, 0.5 \leq x_2 \leq 0.75\}$.
- A basic concept is that the measure function is a continuous mapping between the set X and the set of real numbers.
- Our "Take-Home-Message" is that we can define this measure in a way that yields a probability.

Rings and Sets

- Starting with the simple set $x \in [0,1]$, we want to define a "family" of sets F.
- For now $\{ A_1, A_2, A_3 \} \in F$.



- We are interested in the conditions under which we can define the measure on this set of sets. $\mathfrak{P}$.

Rings and Algebras

- **Definition 1.1 (Rings and Algebras)** A non-empty subset of $\mathfrak{A}$ of $\mathfrak{P}(X)$ is said to be a *ring* if:
  - i $\varnothing$ belongs to $\mathfrak{A}$,
  - ii $A, B \in \mathfrak{A} \Rightarrow A \cup B, A \cap B \in \mathfrak{A}$,
  - iii $A, B, \in \mathfrak{A} \Rightarrow A \setminus B \in \mathfrak{A}$.

  We say that a ring is an *algebra* if $X \in \mathfrak{A}$ (Ambrasio, Da Prato, and Mennucci, p. 2).
- What does this mean? In one sense, the family of subsets completely covers the set of possible sets defined.
- Let us modify our collection of set slighlty $F \supset \{\ A_1, A_2, A_3, \varnothing\}$

σ-Algebra and Borel σ-Algebra

- **Defintion 1.2 (σ-algebras)**. A non-empty subset $\mathfrak{E}$ of $\mathfrak{P}(X)$ is said to be a σ-algebra if:
  - i $\mathfrak{E}$ is an algebra;
  - ii if (A_n) is a sequence of elements of $\mathfrak{E}$ then $\cup_{n=0}^{\infty} A_n \in \mathfrak{E}$.

  (Ambrasio, De Prato, and Mennucci, P.3).
- **Definition 1.3 (Borel σ-algebra)** If $(\mathfrak{E}, d)$ is a metric space, the σ-algebra generated by all open subsets of $\mathfrak{E}$ is called the *Borel σ-algebra* of $\mathfrak{E}$ and it is denoted by $\mathfrak{B}(\mathfrak{E})$. (Ambrasio, De Prato, and Mennucci, p.3).

Additive Measures

- In general measures are additive

$$A, B \in \mathfrak{A}, A \cap B = \varnothing \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B) \tag{11}$$

(Ambrasio, De Prato, and Mennucii, p.4)

- A set function μ is called *σ-additive* if $\mu(\varnothing) = 0$ and for any sequence $(A_n) \subset \mathfrak{A}$ of mutually disjoint sets such that $\cup_n A_n \in \mathfrak{A}$ we have

$$\mu\left(\bigcup_{n=0}^{\infty} A_n\right) = \sum_{i=1}^{\infty} \mu(A_n) \tag{12}$$

A Numeric Example

- Going back to Equation 9 and defining $A_1 = (0, 0.25)$, $A_2 = (0.25, 0.75)$ and $A_3 = (0.75, 1.0)$. We will ignor the endpoints for the time being.
- Based on these definitions $\mu(A_1) = 0.25$, $\mu(A_2) = 0.50$, and $\mu(A_3) = 0.25$.
- Note that the sets are disjoint; hence,

$$\mu\left(\bigcup_{n=1}^{3} A_n\right) = \mu(A_1) + \mu(A_2) + \mu(A_3) = 0.25 + 0.50 + 0.25 = 1.0 \tag{13}$$

- The brief look forward – we are going to define the probability as a measure such that the measures sum to one.

Two-Dimensional Measure

- The algebra is σ-subadditive if

$$\mu(B) \leq \sum_{n=0}^{\infty} \mu(A_n) \tag{14}$$

where $B \in \mathfrak{A}$ and any sequence $(A_n) \subset \mathfrak{A}$ such that $B \subset \cup_n A_n$.

- Using the definition of a measure from Equation 10.

	$(1.00, 0.75)$	$(0.75, 0.50)$	$(0.50, 0.25)$	$(0.25, 0.00)$
$(1.00, 0.75)$	0.0625	0.0625	0.0625	0.0625
$(0.75, 0.50)$	0.0625	0.0625	0.0625	0.0625
$(0.50, 0.25)$	0.0625	0.0625	0.0625	0.0625
$(0.25, 0.00)$	0.0625	0.0625	0.0625	0.0625

- Note that this function is σ-additive and σ-subadditive.