Basic Axioms of Probability: Lecture II

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Hubbard Glacier

- Conover, R.A. (2003) *A First Course in Topology* Dover.

Bloom's Taxonomy

Adapted from
Oregon State University Ecampus

Bloom's Taxonomy Revisited

Use this table as a reference for evaluating and making changes to aligned course activities and assessments (or, where possible, learning outcomes) that account for generative Artificial Intelligence (AI) tool capabilities and distinctive human skills.

All course activities and assessments will benefit from **review** given the capabilities of AI tools; those at the **Remember** and **Analyze** levels may be more likely to need **amendment**.

	DESCRIPTION	AI CAPABILITIES	DISTINCTIVE HUMAN SKILLS
CREATE	Use information to create something new	Suggest a range of alternatives, enumerate potential drawbacks and advantages, describe successful real-world cases	Formulate original solutions incorporating human judgement, collaborate spontaneously
EVALUATE	Examine information and make judgments	Identify pros and cons of various courses of action, develop rubrics	Engage in metacognitive reflection, holistically appraise ethical consequences of alternative courses of action
ANALYZE	Take apart the known and identify relationships	Compare and contrast data, infer trends and themes, compute, predict	Critically think and reason within the cognitive and affective domains, interpret and relate to authentic problems, decisions, & choices
APPLY	Use information in a new (but similar) situation	Make use of a process, model, or method to illustrate how to solve a quantitative inquiry	Operate, implement, conduct, execute, experiment, and test in the real world; apply creativity and imagination to idea & solution development
UNDERSTAND	Grasp meaning of instructional materials	Describe a concept in different words, recognize a related example, translate	Contextualize answers within emotional, moral, or ethical considerations
REMEMBER	Recall specific facts	Recall factual information, list possible answers, define a term, construct a basic chronology	Recall information in situations where technology is not readily accessible

HIGHER-ORDER THINKING SKILLS
LOWER-ORDER THINKING SKILLS

Justifications or development for probabilities

- Frequency approach – A relative count approach – developed from games of chance (aleatory)
- Personal probabilities where individuals anticipate the likelihood of outcomes based on personal information (Leonard Savage and Bruno de Finetti)
- Axiomatic approach where the properties of a probability function are derived from basic conjectures.

Basics of Probability

- Using the example from Bierens Chapter 1: Assume we are interested in the game Texas lotto (similar to Florida lotto).
  - In this game, players choose a set of 6 numbers out of the first 50. Note that the ordering does not count so that 35,20,15,1,5,45 is the same of 35,5,15,20,1,45.
  - How many different sets of numbers can be drawn?
    - First, we note that we could draw any one of 50 numbers in the first draw.
    - However for the second draw we can only draw 49 possible numbers (one of the numbers has been eliminated). Thus, there are 50 x 49 different ways to draw two numbers
- Birens, H.J. 2005. *Introduction to the Mathematical and Statistical Foundations of Econometrics* Cambridge University Press. (uflib.ufl.edu)

- Again, for the third draw, we only have 48 possible numbers left. Therefore, the total number of possible ways to choose 6 numbers out of 50 is

$$\prod_{j=1}^{5}(50-j)=\prod_{k=45}^{50}k=\frac{\prod\limits_{k=1}^{50}k}{\prod\limits_{k=1}^{50-6}k}=\frac{50!}{(50-6)!} \tag{1}$$

- Finally, note that there are 6! ways to draw a set of 6 numbers (you could draw 35 first, or 20 first, ...). Thus, the total number of ways to draw an unordered set of 6 numbers out of 50 is

$$\left(\begin{array}{c}50\\6\end{array}\right)=\frac{50!}{6!\,(50-6)!}=15,890,700 \tag{2}$$

- This is a combinatorial. It also is useful for binomial arithmetic:

$$(a+b)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^k b^{n-k} \tag{3}$$

Definitions

- *Sample Space*: The set of all possible outcomes. In the Texas lotto scenario, the sample space is all possible 15,890,700 sets of 6 numbers which could be drawn.
- *Event*: A subset of the sample space. In the Texas lotto scenario, possible events include single draws such as 35,20,15,1,5,45 or complex draws such as all possible lotto tickets including 35,20,15. Note that this could be 35,20,15,1,2,3, 35,20,15,1,2,4, ...
- *Simple Event*: An event which cannot be a union of other events. In the Texas lotto scenario, this is a single draw such as 35,20,15,1,5,45.
- *Composite Event*: An event which is not a simple event.

Counting Techniques

- Simple events with equal probabilities

$$P\left[A\right] = \frac{n(A)}{n(S)} \tag{4}$$

  the probability of event A is simply the number possible occurrences of A divided by the number of possible occurrences in the sample.

- Definition 2.3.1 The number of permutations of taking r elements from n elements is a number of distinct ordered sets consisting of r distinct elements which can be formed out of a set of n distinctive elements and is denoted P_r^n.

- The first point to consider is that of factorials. For example, if you have two objects A and B, how many different ways are there to order the object? Two:

$$\{\ A, B\}\ \text{ or } \{\ B, A\} \tag{5}$$

If you have three orderings how many ways are there to order the objects? Six:

$$\begin{array}{c}\{\ A, B, C\}\ \{\ A, C, B\}\ \{\ B, A, C\}\ \{\ B, C, A\}\ ,\\ \{\ C, A, B\}\ \text{ or } \{\ C, B, A\}\end{array} \tag{6}$$

The sequence then becomes two objects can be drawn in two sequences, three objects can be drawn in six sequences (2 x 3). By inductive proof, four objects can be drawn in 24 sequences (6 x 4).

- The total possible number of sequences is then for n objects is $n!$ defined as:

$$n! = n\,(n-1)\,(n-2)\ldots 1 \tag{7}$$

- Theorem 2.3.1:

$$P_r^n = \frac{n!}{(n-r)!} \tag{8}$$

- Definition 2.3.2: The number of combinations of taking r elements from n elements is the number of distinct sets consisting of r distinct elements which can be formed out of a set of n distinct elements and is denoted C_r^n.

$$C_r^n = \frac{n!}{(n-r)!r!} \tag{9}$$

Expected Utility – The von Neumann - Morgenstern Formulation

- vonNeumann, J and O. Morgenstern (1953). *Theory of Games and Economic Behavior* Princeton University Press.
- The basic conjecture is that economic agents choose between two risky alternatives in a way that maximizes their expected utility.
    - A typical utility function is the Power Utility function

$$U\left(Y\right)=\frac{Y^{1-r}}{r} \tag{10}$$

where $r>0$ represents the relative risk aversion coefficient.

A Choice – Leonard Savage

- To build Savage's concept, suppose that we have two possible *events* or outcomes on *sample space* each of which has different implications

	Action	
Outcome	A	B
1	100,000	125,000
2	150,000	130,000

- Suppose that I assume that two individuals have the same risk aversion coefficient (i.e., $r = 1 \Rightarrow U(Y) = \ln(Y)$). For each individual, the value of the investment becomes

$$\begin{aligned} E_{i\mathrm{A}} &= P_{i,1}\ln(100,000) + P_{i,2}\ln(150,000) \\ E_{i\mathrm{B}} &= P_{i,1}\ln(125,000) + P_{i,2}\ln(130,000) \end{aligned} \tag{11}$$

Choice Implies Probability

- In this case, suppose that two individuals make different choices – individual a chooses alternative A while individual b chooses alternative B. The only explanation is that the individuals have different perceived probabilities – $P_{a,1} \neq P_{b,1}$ (and by summing up restrictions $P_{a,2} \neq P_{b,2}$).

Leonard Savage - Foundations of Statistics

- Savage, L. (1972) *The Foundations of Statistics* Dover Publications, New York.
- Three main classes of view of the interpretation of probability:
  - **Objectivistic –** some repetitive events prove to be in reasonably close agreement with the mathematical concpet of independently repreated random events.
  - **Personalistic –** probability measures the confidence that a particular individual has in the truth of a particular proposition.
  - **Necessary –** probability measures can be logically derived from a set of propositions apart from human opinioin (i.e., Kolgomorov)

State of the Egg and Action

	State	
Act	Good	Rotten
Break into bowl	six-egg omelet	no omelet and five good eggs spoilt
Break egg into saucer	six-egg omelet and a saucer to wash	five-egg omelet and a saucer to wash
Throw away	five-egg omelet, and one good egg destroyed	five-egg omelet

- Each action interacts with the state – we will call the first action f - break the egg into the bowl. The objective value of f is a function of the probaility of the states

$$f \approx P_1 f\left(\text{good}\right) + P_2 f\left(\text{rotten}\right). \tag{12}$$

The decision is implicitly compared with the objective value of another action g say break the egg into the saucer.

Sure Thing Principle

- **P1** The relationship $\precsim$ is a simple ordering among acts [Savage p. 18.]
- **Theorem 1** If F is a finite set of sets, there exists f and h in F such that for all g in F
$$f \precsim g \precsim h. \tag{13}$$
- **Sure Thing Principle**: If the person would not prefer f to g, either knowing that the event B obtained, or not knowing that the the event $\sim B$ obtained, then he does not prefer f to g.

Sure Thing Figure

THE SURE-THING PRINCIPLE

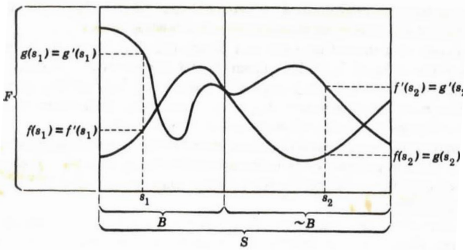

Figure 1

- **P2** If f, g, and f', g' are such that:
  1. in $\sim B$, f agrees with g and f' agrees with g',
  2. in B, f agrees with f' and g agrees with g',
  3. $f \precsim g$;
  - Then $f' \precsim g'$.
- A finite set of events B_i is a partition of B, and $f \precsim g$ given $B_i \cap B_j = \varnothing$ for $i \neq j$ and $\cup_i B_i = B$
- **Theorem 2** If B_i is a partition of B, and $f \precsim g$ given B_i for each i, then $f \precsim g$ given B. If, in addition, $f \prec g$, given B_j for at least one j, $f \prec g$ given B.
- A relationship $\precsim$ between events is a qualitative probability; if and only if, for all events B, C, and D,
  1. $\precsim$ is a simple ordering,
  2. $B \precsim C$, if and only if $B \cup D \precsim C \cup D$, provided $B \cap D = C \cap D = \varnothing$,
  3. $\varnothing \precsim B$, $\varnothing \precsim S$.