

Empirical Maximum Likelihood and Method of Moment Estimation: Lecture XXa

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I. Gamma Distribution Function

$$f(x|\alpha, \beta) = \frac{x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right)}{\Gamma(\alpha) \beta^\alpha} \quad (1)$$

A. The likelihood function for a sample of random variables distributed gamma can be expressed as

$$L(x_1, x_2, \dots, x_n|\alpha, \beta) = \frac{1}{(\Gamma(\alpha) \beta^\alpha)^n} \prod_{i=1}^n x_i^{\alpha-1} \exp\left[-\frac{x_i}{\beta}\right]. \quad (2)$$

B. Taking the logarithm of the likelihood function

$$\begin{aligned} \ln(L(x_1, x_2, \dots, x_n|\alpha, \beta)) &= -n \ln(\Gamma(\alpha)) - n\alpha \ln(\beta) \\ &\quad + (\alpha - 1) \sum_{i=1}^n \ln(x_i) - \frac{1}{\beta} \sum_{i=1}^n x_i \end{aligned} \quad (3)$$

C. Note that if we define two statistics

$$\begin{aligned} T_1 &= \sum_{i=1}^n \ln(x_i) \\ T_2 &= \sum_{i=1}^n x_i \end{aligned} \quad (4)$$

The sample likelihood function becomes

$$\ln(L(x_1, x_2, \dots, x_n | \alpha, \beta)) = -n \ln(\Gamma(\alpha)) - n\alpha \ln(\beta) + (\alpha - 1)T_1 - \frac{1}{\beta}T_2 \quad (5)$$

D. Code to estimate the parameters of the gamma in R is then

```
dta <- read.table("Gamma Data 2007.dta")

t1 <- sum(dta)
t2 <- sum(log(dta))

fr <- function(b) { nrow(dta)*lgamma(b[1])+nrow(dta)*b[1]*log(b[2]) -
  (b[1]-1)*t2+1/b[2]*t1 }

gfr <- function(b) { cbind(nrow(dta)*digamma(b[1])+nrow(dta)*log(b[2])-t2,
  nrow(dta)*b[1]/b[2]-1/b[2]^2*t1) }

res <- optim(c(1.0,1.0),fr,gfr,method="BFGS")
```

E. To compute the variance of the estimates

```
z <- hessian(fr,res$par,method="Richardson")

print(z)

zinv <- solve(z)
$
```

F. This code results in estimates $\alpha = 0.6947$ and $\beta = 1.2312$. And a variance matrix

$$zinv = V(\theta) = \begin{bmatrix} 0.0140 & -0.0248 \\ -0.0248 & 0.0876 \end{bmatrix} \quad (6)$$

II. Method of MOments Estimator

A. We know that the first four moments of the gamma distribution are

$$\begin{aligned}\mu_1(0) &= \alpha\beta \\ \mu_2^*(0) &= \alpha\beta^2 \\ \mu_3^*(0) &= \frac{2}{\sqrt{\alpha}} \\ \mu_4^*(0) &= \frac{6}{\alpha}\end{aligned}\tag{7}$$

B. Code for Exact Method of Moments

```
dta <- read.table("Gamma Data 2007.dta")
t1 <- mean(dta)
t2 <- var(dta)

fr <- function(b) { (b[1]*b[2] - t1)^2 + (b[1]*b[2]^2 - t2)^2 }

gfr <- function(b) { cbind(2*(b[1]*b[2] - t1)*b[2] + 2*(b[1]*b[2]^2 - t2)*b[2]^2,
    2*(b[1]*b[2] - t1)*b[1] + 4*(b[1]*b[2]^2 - t2)*b[1]*b[2]) }

res <- optim(c(3.0,1.0),fr,gfr,method="BFGS")
```

C. Actually the code minimized a squared error term

$$Q(\alpha, \beta) = (E[x] - \alpha\beta)^2 + (E[x - E[x]] - \alpha\beta^2)^2\tag{8}$$

D. Overidentified Method of Moments Estimator

$$\begin{aligned}Q(\alpha, \beta) &= (E[x] - \alpha\beta)^2 + (E[x - E[x]] - \alpha\beta^2)^2 \\ &+ \left(E[x - E[x]]^3 - \frac{2}{\sqrt{\alpha}}\right)^2 + \left(E[x - E[x]]^4 - \frac{6}{\alpha}\right)^2\end{aligned}\tag{9}$$

```
dta <- read.table("Gamma Data 2007.dta")
t1 <- mean(dta)
t2 <- var(dta)
t3 <- mean((dta-t1)^3)
t4 <- mean((dta-t1)^4)

fr <- function(b) { (b[1]*b[2] - t1)^2 + (b[1]*b[2]^2 - t2)^2 + (2/sqrt(b[1]) - t3)^2 +
    (6/b[1] - t4)^2 }
```

```
gfr <- function(b) { cbind(2*(b[1]*b[2]- t1)*b[2] + 2*(b[1]*b[2]^2 -  
    t2)*b[2]^2 -2*(2/sqrt(b[1])- t3)/(b[1])^(3/2) - 12*(6/b[1] - t4)/b[1]^2,  
    2*(b[1]*b[2] - t1)*b[1] + 4*(b[1]*b[2]^2 - t2)*b[1]*b[2]) }  
  
res <- optim(c(3.0,2.0),fr,gfr,method="BFGS")
```