

Empirical Examples of the Central Limit Theorem: Lecture XVI

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I. Back to Asymptotic Normality

A. The characteristic function of a random variable X is defined as

$$\begin{aligned}\phi_X(t) &= \mathbb{E}[e^{itX}] = \mathbb{E}[\cos(tX) + i\sin(tX)] \\ &= \mathbb{E}[\cos(tX)] + i\mathbb{E}[\sin(tX)].\end{aligned}\tag{1}$$

Note that this definition parallels the definition of the moment-generating function

$$M_X(t) = \mathbb{E}[e^{tX}]\tag{2}$$

1. Like the moment-generating function there is a one-to-one correspondence between the characteristic function and the distribution of random variable. Two random variables with the same characteristic function are distributed the same.
2. The characteristic function of the uniform distribution function of the uniform distribution function is

$$\phi_X(t) = e^{it} - 1\tag{3}$$

The characteristic function of the Normal distribution function is

$$\phi_X(t) = \exp\left(it\mu - \frac{\sigma^2 t^2}{2}\right)\tag{4}$$

The Gamma distribution function

$$f(X) = \frac{a^r x^{r-1} e^{-\alpha X}}{\Gamma(r)} \quad X \in (0, \infty) \quad (5)$$

which implies the characteristic function

$$\phi_X(t) = \frac{1}{\left(1 - \frac{it}{\alpha}\right)^r} \quad (6)$$

B. Taking a Taylor series expansion of around the point $t = 0$ yields

$$\phi_Z(t) = \phi_Z(0) + \frac{1}{1!} \phi'_Z(0) t + \frac{1}{2!} \phi''_Z(0) t^2 + o(t^2) \quad (7)$$

$$\ni Z = \frac{X - \mu}{\sigma\sqrt{n}}$$

To work on this expression we note that

$$\phi_X(0) = 1 \quad (8)$$

For any random variable X , and

$$\phi_X^{(k)}(0) = i^k E(X^k) \quad (9)$$

Putting these two results into the second-order Taylor series expansion

$$1 + \frac{E(Z)}{i} t + \frac{E(Z^2)}{i^2} \frac{t^2}{2} + o(t^2) = 1 - \frac{t^2}{2} + o(t^2) \quad (10)$$

$$\ni E(Z) = 0, \quad E(Z^2) = 1$$

Thus,

$$\begin{aligned}
 \phi_z(t) &= \phi_z(0) + \frac{1}{1!}\phi'_z(0)t + \frac{1}{2!}\phi''_z(0)t^2 + o(t^2) \\
 &= 1 + \frac{E(Z)}{i}t + \frac{E(Z)t^2}{i^2 \cdot 2} + o(t^2) \\
 &= 1 - \frac{t^2}{2} + o(t^2) \Leftrightarrow 1 - \frac{t^2}{2} = E(e^{iy}) \ni y \sim N(0, 1)
 \end{aligned} \tag{11}$$

II. Wrapping Up Loose Ends

A. Application of Holder's Inequality.

1. Holder's inequality

$$|E[XY]| \leq E|XY| \leq (E|X|^p)^{1/p} (E|Y|^q)^{1/q} \tag{12}$$

2. **Example 4.7.1:** If X and Y have means μ_X , μ_Y and variances σ_X^2 , σ_Y^2 , respectively, we can apply the Cauchy-Schwartz Inequality (Holders inequality with $p = q = 1/2$) to get

$$E|(X - \mu_X)(Y - \mu_Y)| \leq \left\{ E[(X - \mu_X)^2]^{1/2} \right\} \left\{ E[(Y - \mu_Y)^2] \right\}^{1/2} \tag{13}$$

Squaring both sides and substituting for variances and covariances yields

$$(\text{Cov}(X, Y))^2 \leq \sigma_X^2 \sigma_Y^2 \tag{14}$$

Which implies that the absolute value of the correlation coefficient is less than one.

C. Application of Chebychev's Inequality

1. Chebychev's Inequality: Let X be a random variable and let $g(X)$ be a nonnegative function. Then, for any $r > 0$

$$P[g(X) \geq r] \leq \frac{rE[g(X)]}{r} \tag{15}$$

2. **Example 4.7.3:** The most widespread use of Chebychev's Inequality involves means and variances. Let $g(x) = (x - \mu)^2 / \sigma^2$, where $\mu = E[X]$ and $\sigma^2 = V(X)$. Let $r = t^2$

$$P\left(\frac{(X - \mu)^2}{\sigma^2} \geq t^2\right) \leq \frac{1}{t^2} E\left[\frac{(X - \mu)^2}{\sigma^2}\right] = \frac{1}{t^2} \quad (16)$$

Since

$$\frac{(X - \mu)^2}{\sigma^2} \geq t^2 \Rightarrow (X - \mu)^2 \geq \sigma^2 t^2 \Rightarrow |X - \mu| \geq \sigma t \quad (17)$$

$$P(|X - \mu| \geq \sigma t) \leq \frac{1}{t^2}$$

Letting $t = 2$, this becomes

$$P(|X - \mu| \geq 2\sigma) \leq \frac{1}{4} = 0.25 \quad (18)$$

However, this inequality may not say much, since for the normal distribution

$$P(|X - \mu| \geq 2\sigma) = 2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x - \mu)^2}{2\sigma^2}\right] dx = 2 \times (0.0227) = 0.0455 \quad (19)$$

Thus, the actual probability under the standard normal distribution is much smaller than the Chebychev Inequality.

- D. Casella and Berger offer a proof of the Central Limit Theorem (Theorem 5.3.3) based on the moment generating function instead of the characteristic function. However, they note that the proof using the characteristic function is a stronger result.

III. Normal Approximation of the Binomial

- A. Starting from the Binomial distribution function

$$b(n, r, p) = \frac{n!}{(n - r)!r!} p^r (1 - p)^{n-r} \quad (20)$$

- B. First, assume that $n = 10$ and $p = 0.5$. The probability of $r \leq 3$ is

$$P(r \leq 3) = b(10, 0, 0.5) + b(10, 1, 0.5) + b(10, 3, 0.5) = 0.1719 \quad (21)$$

Note that this distribution has a mean of 5 and a variance of 2.5. Given this we can compute

$$z^* = \frac{3 - 5}{\sqrt{2.5}} = -1.265 \quad (22)$$

Integrating the standard normal distribution function from negative infinity to -1.265 yields

$$P(z^* \leq -1.265) = \int_{-\infty}^{-1.265} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right] dz = 0.1030. \quad (23)$$

- C. Expanding the sample size to 20 and examining the probability that $r \leq 6$ yields

$$P(r \leq 6) = \sum_{i=0}^6 b(20, i, 0.5) = 0.0577 \quad (24)$$

This time the mean of the distribution is 10 and the variance is 5. The resulting $z^* = -1.7889$. The integral of the normal distribution function from negative infinity to -1.7889 is .0368.

- D. As the sample size declines the binomial probability approaches the normal probability.