Binomial and Normal Random Variables

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I. Bernoulli Random Variables

A. The Bernoulli distribution characterizes the coin toss. Specifically, there are two events $X = 0, 1$ with $X = 1$ occurring with probability p. The probability distribution function $P[X]$ can be written as

$$P[X] = p^x (1-p)^x \tag{1}$$

B. Next, we need to develop the probability of where both and are identically distributed. If the two events are independent, the probability becomes

$$\begin{aligned} P[X,Y] = P[X]\,P[Y] &= p^x p^y (1-p)^{1-x} (1-p)^{1-y} \\ &= p^{x+y} (1-p)^{2-x-y} \end{aligned} \tag{2}$$

C. Now, this density function is only concerned with three outcomes $Z = X + Y = \{\ 0, 1, 2\}$.

1. There is only one way each for $Z = 0$ or $Z = 2$.
    a) Specifically for $Z = 0$, $X = 0$ and $Y = 0$.
    b) Similarly, for $Z = 2$, $X = 1$ and $Y = 1$.
2. However, for $Z = 1$ either $X = 1$ and $Y = 0$ or $X = 0$ and $Y = 1$. Thus, we can derive

$$P[Z=0] = p^0 (1-p)^{2-0}$$

$$\begin{aligned} P[Z=1] &= P[X=1, Y=0] + P[X=0, Y=1] \\ &= p^{1+0}(1-p)^{2-1-0} + p^{0+1}(1-p)^{2-0-1} \\ &= 2p^1(1-p)^1 \end{aligned} \tag{3}$$

$$P[Z=2] = p^2 (1-p)^0$$

D. Next we expand the distribution to three independent Bernoulli events where $Z = W + X + Y = \{\ 0, 1, 2, 3\}\ $.

$$\begin{aligned} P[Z] &= P[W, X, Y] \\ &= P[W]\,P[X]\,P[Y] \\ &= p^w p^x p^y (1-p)^{1-w}(1-p)^{1-x}(1-p)^{1-y} \\ &= p^{w+x+y}(1-p)^{3-w-x-y} \\ &= p^z (1-p)^{3-z} \end{aligned} \tag{4}$$

1. Again, there is only one way for $Z = 0$ and $Z = 3$.
2. However, there are now three ways for $Z = 1$ or $Z = 2$. Specifically, $Z = 1$ if $W = 1$, $X = 1$ or $Y = 1$. In addition, $Z = 2$ if $W = 1$, $X = 1$, and $Y = 0$, or if $W = 1$, $X = 0$, and $Y = 1$, or if $W = 0$, $X = 1$, and $Y = 1$.
3. Thus the general distribution function for Z can now be written as

$$
\begin{aligned}
P\left[Z=0\right] &= p^0\left(1-p\right)^{3-0}\\
P\left[Z=1\right] &= p^{1+0+0}\left(1-p\right)^{3-1-0-0}+p^{0+1+0}\left(1-p\right)^{3-0-1-0}+\\
&\quad p^{0+0+1}\left(1-p\right)^{3-0-0-1}=3p^1\left(1-p\right)^2\\
P\left[Z=2\right] &= p^{1+1+0}\left(1-p\right)^{3-1-1-0}+p^{1+0+1}\left(1-p\right)^{3-1-0-1}+\\
&\quad p^{0+1+1}\left(1-p\right)^{3-0-1-1}=3p^2\left(1-p\right)^1\\
P\left[Z=3\right] &= p^3\left(1-p\right)^0
\end{aligned}
\tag{5}
$$

II. Binomial Random Variables

A. Based on this development, the binomial distribution can be generalized as the sum of n Bernoulli events

$$
\begin{aligned}
P\left[Z=r\right] &= \begin{pmatrix} n\\ r\end{pmatrix} p^r\left(1-p\right)^{n-r}\\
C_r^n &= \begin{pmatrix} n\\ r\end{pmatrix} = \frac{n!}{r!\left(n-r\right)!}.
\end{aligned}
\tag{6}
$$

where C_r^n is the combinatorial of n and r. Graphically the combinatorial can be depicted as Pascal's triangle in Figure 1. To develop the relationship between the combinatorial and the binomial function, consider the polynomial form

$$
\begin{aligned}
\left(a+b\right)^1 &= a+b\\
\left(a+b\right)^2 &= a^2+2ab+b^2\\
\left(a+b\right)^3 &= a^3+3a^2b+3ab^2+b^3\\
\left(a+b\right)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4
\end{aligned}
\tag{7}
$$

The general form of the polynomial can then be written as

$$
\left(a+b\right)^n=\sum_{r=1}^{n} C_r^n a^r b^{n-r} \tag{8}
$$

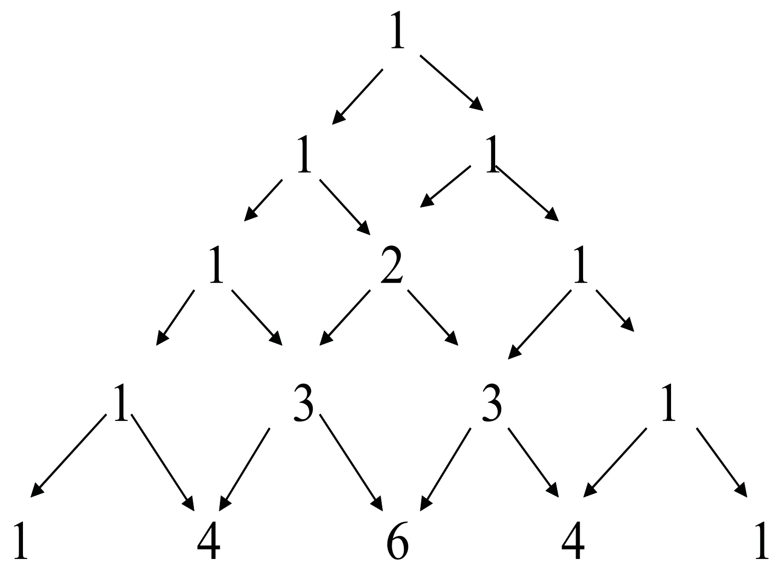

Figure 1: Pascal's Triangle

As a first step, consider changing the polynomial from $a+b$ to $a-b$. Hence, $(a-b)^n$ can be written as

$$\begin{aligned}(a-b)^n &= (a+(-1)\,b)^n \\ &= \sum_{r=1}^{n} C_r^n a^r (-1)^{n-4} b^{n-4} \\ &= \sum_{r=1}^{n} (-1)^{n-r} C_r^n a^r b^{n-4}\end{aligned} \tag{9}$$

1. This sequence can be linked to our discussion of the Bernoulli system by letting $a=p$ and $b=1-p$. Thus, the binomial distribution $X \sim B(n,p)$ can be written as

$$P(X=k) = C_k^n p^k (1-p)^{n-k} \tag{10}$$

2. For the case above, $n=3$. The distribution function (ignoring the constants) can be written as

$$\begin{aligned}
P[Z=0] &= \begin{pmatrix} 3 \\ 0 \end{pmatrix} p^0 (1-p)^{3-0} \\
P[Z=1] &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} p^1 (1-p)^{3-1} \\
P[Z=2] &= \begin{pmatrix} 3 \\ 2 \end{pmatrix} p^2 (1-p)^{3-2} \\
P[Z=3] &= \begin{pmatrix} 3 \\ 3 \end{pmatrix} p^3 (1-p)^{3-3}
\end{aligned} \tag{11}$$

B. Next recalling Theorem 4.1.6: $\mathrm{E}[aX+bY] = a\mathrm{E}[X] + b\mathrm{E}[Y]$, the expectation of the binomial distribution function can be recovered from the Bernoulli distributions

$$\begin{gathered}
\mathrm{E}[X] = \sum_{k=0}^{n} C_k^n p^k (1-p)^{n-k} k \\
= \sum_{i=1}^{n} \left[\sum_{X_i} p^{X_i} (1-p)^{1-X_i} X_i \right] \\
= \sum_{i=1}^{n} \left[p^1 (1-p)^0 (1) + p^0 (1-p)^1 (0) \right] \\
= \sum_{i=1}^{n} p = np
\end{gathered} \tag{12}$$

C. In addition, by Theorem 4.3.3

$$V\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} V(X_i). \tag{13}$$

Thus, variance of the binomial is simply the sum of the variances of the Bernoulli distributions or n times the variance of a single Bernoulli distribution

$$\begin{gathered}
V(X) = \mathrm{E}\left[X^2\right] - (\mathrm{E}[X])^2 \\
= \left[p^1 (1-p)^0 \left(1^2\right) + p^0 (1-p)^1 \left(0^2\right) \right] - p^2 \\
= p - p^2 = p(1-p) \\
V\left(\sum_{i=1}^{n}\right) = np(1-p)
\end{gathered} \tag{14}$$