Mean and Higher Moments

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I. Expected Value

A. **Definition 4.1.1**: Let X be a discrete random variable taking the value x with probability $P(x_i)$, $i = 1, 2, \cdots$. Then the expected value (expectation or mean) of X, denoted $\mathrm{E}[\mathrm{X}]$, is defined to be $\mathrm{E}[\mathrm{X}] = \sum_{\mathrm{i}=1}^{\infty} \mathrm{x_i P(x_i)}$ if the series converges absolutely.

1. We can write $\mathrm{E}[\mathrm{X}] = \sum_{+} \mathrm{x_i P(x_i)} + \sum_{-} \mathrm{x_i P(x_i)}$ where in the first summation we sum for i such that $x_i > 0$ and in the second summation we sum for i such that $x_i < 0$.
2. If $\sum_{+} x_i P(x_i) = \infty$ and $\sum_{-} x_i P(x_i) = \infty$ then $\mathrm{E}[\mathrm{X}]$ does not exist.
3. If $\sum_{+} x_i P(x_i) = \infty$ and $\sum_{-} x_i P(x_i)$ is finite then we say $\mathrm{E}[\mathrm{X}] = \infty$.
4. If $\sum_{-} P(x_i) = -\infty$ and $\sum_{+} x_i P(x_i)$ is finite then we say that $\mathrm{E}[\mathrm{X}] = -\infty$.

B. A practical application:

1. Given that each face of the die is equally likely, what is the expected value of the role of the die?
2. What is the expected value of a two-die role?

C. Expectation has several applications in risk theory. In general, the expected value is the value we expect to occur. For example, if we assume that the crop yield follows a binomial distribution as depicted in Figure 1, the expected return on the crop given that the price is \$3 and the cost per acre is \$40, becomes \$ 95 per acre as demonstrated in Table 3.

Table 1: Expected Value of a Single Die Role

Number	Probability	$x_i P(x_i)$
1	0.167	0.167
2	0.167	0.333
3	0.167	0.500
4	0.167	0.667
5	0.167	0.833
6	0.167	1.000
Total		3.500

Table 2: Expected Value of a Two-Die Role

Die 1	Die 2	Number	$x_i P(x_i)$	Die 1	Die 2	Number	$x_i P(x_i)$
1	1	2	0.056	1	4	5	0.139
2	1	3	0.083	2	4	6	0.167
3	1	4	0.111	3	4	7	0.194
4	1	5	0.139	4	4	8	0.222
5	1	6	0.167	5	4	9	0.250
6	1	7	0.194	6	4	10	0.278
1	2	3	0.083	1	5	6	0.167
2	2	4	0.111	2	5	7	0.194
3	2	5	0.139	3	5	8	0.222
4	2	6	0.167	4	5	9	0.250
5	2	7	0.194	5	5	10	0.278
6	2	8	0.222	6	5	11	0.306
1	3	4	0.111	1	6	7	0.194
2	3	5	0.139	2	6	8	0.222
3	3	6	0.167	3	6	9	0.250
4	3	7	0.194	4	6	10	0.278
5	3	8	0.222	5	6	11	0.306
6	3	9	0.250	6	6	12	0.333
				Total			7.000

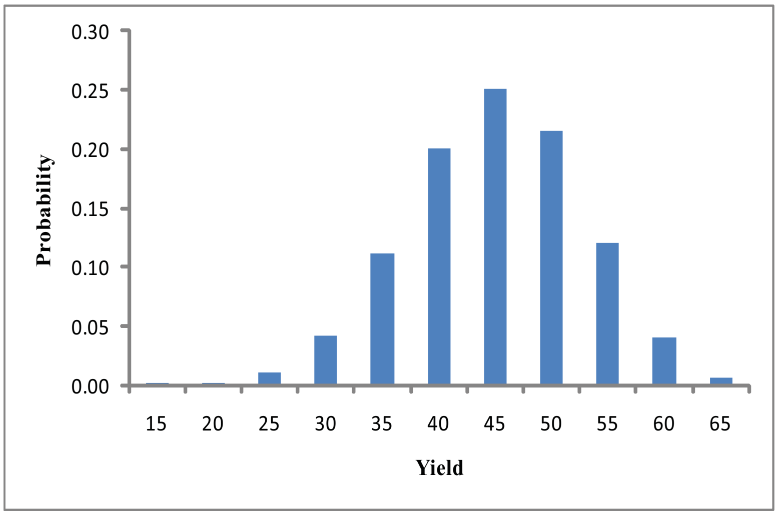

Figure 1: Wheat Yield Density Function

Table 3: Expected Return on an Acre of Wheat

X	$P[X]$	$x_i P[x_i]$	$(p_x x_i - C) P(x_i)$
15	0.0001	0.0016	0.0005
20	0.0016	0.0315	0.0315
25	0.0106	0.2654	0.3716
30	0.0425	1.2740	2.1234
35	0.1115	3.9017	7.2460
40	0.2007	8.0263	16.0526
45	0.2508	11.2870	23.8282
50	0.2150	10.7495	23.6490
55	0.1209	6.6513	15.1165
60	0.0403	2.4186	5.6435
65	0.0060	0.3930	0.9372
Total		45.0000	95.0000

D. In the parlance of risk theory, the expected value of the wheat crop is termed the actuarial or fair value of the game. It is the value that a risk neutral individual would be willing to pay for the bet Moss 2010.

E. Another point about this value, it is sometimes called the population mean as opposed to the sample mean. Specifically, the sample mean is an observed quantity based on a sample drawn from the random generating function. The sample mean is defined as

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \tag{1}$$

Table 4 presents a sample of 20 observations drawn from the theoretical distribution above, note that sample mean for yield is smaller than the population mean (33.75 for the sample mean versus 45.00 for the population mean). It follows that the sample mean for profit is smaller than the population mean for profit.

F. Another insight from the expected value and gambling is the Saint Petersburg paradox. The Saint Petersburg paradox involves the valuation of gambles with an infinite value. The simplest form of the paradox involves the value of a series of coin flips. Specifically what is the expected value of a bet that pays off \$2 for each consecutive head? For example, if the series of coin flips is HHHT, the payoff is \$8. In theory, the expected value of this bet is infinity, but no one is willing to pay an infinite price.

$$\mathrm{E}\left[\mathrm{G}\right] = \sum_{\mathrm{i}=1}^{\infty} 2^{\mathrm{i}}2^{-\mathrm{i}} = \sum_{\mathrm{i}=1}^{\infty} 1 = \infty \tag{2}$$

This unwillingness to pay an infinite price for the gamble led to expected utility theory.

G. **Definition 4.1.2.** Let X be a continuous random variable with density $f(x)$. Then, the expected value of X, denoted $\mathrm{E}[\mathrm{X}]$, is defined to be $\mathrm{E}[\mathrm{X}] = \int_{-\infty}^{\infty} \mathrm{xf}(\mathrm{x})\,\mathrm{dx}$ if the integral is absolutely convergent.

   * If $\int_0^{\infty} xf(x)\,dx = \infty$ and $\int_{-\infty}^{0} xf(x)\,dx = \infty$, we say that the expectation does not exist.

Table 4: Sample of Yields and Profits

Observation	Yield	Profit
1	40	80
2	40	80
3	40	80
4	50	110
5	50	110
6	45	95
7	35	65
8	25	35
9	40	80
10	50	110
11	30	50
12	35	65
13	40	80
14	25	35
15	45	95
16	35	65
17	35	65
18	40	80
19	30	50
20	45	95
Mean	38.75	76.25

- If $\int_0^\infty xf(x)\,dx = \infty$ and $\int_{-\infty}^0 xf(x)\,dx$ is finite then $\mathrm{E}[\mathrm{X}] = \infty$.
- If $\int_{-\infty}^0 xf(x)\,dx = \infty$ and $\int_0^\infty xf(x)\,dx$ is finite, then we write $\mathrm{E}[\mathrm{X}] = -\infty$.

H. **Theorem 4.1.3.** Let (X,Y) be a bivariate discrete random variable taking value (x_i, y_i) with probability $P(x_i, y_i)$, $i, j = 1, 2, \cdots$ and let $\phi(x_i, y_i)$ be an arbitrary function. Then

$$\mathrm{E}[\phi(\mathrm{X},\mathrm{Y})] = \sum_{\mathrm{i}=1}^{\infty}\sum_{\mathrm{j}=1}^{\infty}\phi(\mathrm{x_i},\mathrm{y_i})\,\mathrm{P}(\mathrm{x_i},\mathrm{y_i}) \tag{3}$$

I. **Theorem 4.1.4.** Let (X,Y) be a bivariate continuous random variable with joint density function $f(x,y)$, and let $\phi(x,y)$ be an arbitrary function. Then

$$\mathrm{E}[\phi(\mathrm{X},\mathrm{Y})] = \int_{-\infty}^{\infty}\int_{-\infty}\infty\phi(\mathrm{x},\mathrm{y})\,\mathrm{f}(\mathrm{x},\mathrm{y})\,\mathrm{dxdy} \tag{4}$$

J. **Theorem 4.1.5.** If α is a constant, $\mathrm{E}[\alpha] = \alpha$.

K. **Theorem 4.1.6.** If X and Y are random variables and α and β are constants,

$$\mathrm{E}[\mathrm{XY}] = \alpha\mathrm{E}[\mathrm{X}] + \beta\mathrm{E}[\mathrm{Y}] \tag{5}$$

L. **Theorem 4.1.7.** If X and Y are independent random variables, $\mathrm{E}[\mathrm{XY}] = \mathrm{E}[\mathrm{x}]\,\mathrm{E}[\mathrm{Y}]$.

M. The last series of theorems are important to simplify decision making under risk.

1. In the crop example we have

$$\pi = pX - C \tag{6}$$

where π is profit, p is the price of the output, X is the yield level and C is the cost per acre. The distribution of profit along with its expected value is dependent on the distribution of p , X , and C. In the example above, we assume that p and C are constant at $\tilde{p}$ and $\tilde{C}$. The expected value of profit is then:

$$E\left[\pi=\phi\left(p,X,C\right)\right]=E\left[pX-C\right]=\tilde{p}E\left[X\right]-\tilde{c} \tag{7}$$

2. As a first step, assume that cost is a random variable

$$E\left[\pi=\phi\left(p,X,C\right)\right]=E\left[pX-C\right]=\tilde{p}E\left[x\right]-E\left[C\right] \tag{8}$$

3. Next, assume that price and yield are random, but cost is constant

$$E\left[\pi=\phi\left(p,X,C\right)\right]=E\left[pX\right]-\tilde{c}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}pxf\left(x,y\right)dx\,dp \tag{9}$$

By assuming that p and X are independent (firm level assumptions)

$$E\left[\pi=\phi\left(p,X,C\right)\right]=E\left[p\right]E\left[X\right]-\tilde{c} \tag{10}$$

II. Moments

A. Another frequently used function of random variables is the moments of the distribution function

$$\mu_r\left(X\right)=E\left[X^r\right]=\int_{-nfty}^{\infty}x^r f\left(x\right)dx \tag{11}$$

where r is a nonnegative integer. From this definition, it is obvious that the mean is the first moment of the distribution function. The second moment is defined as

$$\mu_2\left(X\right)=E\left[X^2\right]=\int_{-\infty}^{\infty}x^2 f\left(x\right)dx \tag{12}$$

B. The higher moments can similarly be represented as moments around the mean or central moments

$$\tilde{\mu}_r\left(X\right)=E\left[X-E\left[X\right]\right]^r \tag{13}$$

C. Examples, the first, second, third and fourth moments of the uniform distribution can be defined as

$$\begin{array}{c}\mu_1(X)=\int_0^1 x(1)\,dx=\frac{1}{2}\left(x^2\right|_0^1=\frac{1}{2}\\ \mu_2(X)=\int_0^1 x^2dx=\frac{1}{3}\left(x^3\right|_0^1=\frac{1}{3}\\ \mu_3(X)=\int_0^1 x^3dx=\frac{1}{4}\left(x^4\right|_0^1=\frac{1}{4}\\ \mu_4(X)=\int_0^1 x^4dx=\frac{1}{5}\left(x^5\right|_0^1=\frac{1}{4}\end{array}\tag{14}$$

D. **Definition 4.2.1.** The second central moment of the distribution defines the variance of the distribution

$$V(X)=\mathrm{E}\left[\mathrm{X}-\mathrm{E}[\mathrm{x}]\right]^2=\mathrm{E}\left[\mathrm{X}^2\right]-\mathrm{E}[\mathrm{X}]^2\tag{15}$$

The last equality is derived by

$$\begin{array}{c}\mathrm{E}\left[\mathrm{X}-\mathrm{E}[\mathrm{X}]\right]^2=\mathrm{E}\left[(\mathrm{X}-\mathrm{E}[\mathrm{X}])(\mathrm{X}-\mathrm{E}[\mathrm{X}])\right]\\ =\mathrm{E}\left[\mathrm{X}^2-2\mathrm{X}\mathrm{E}[\mathrm{X}]+\mathrm{E}[\mathrm{X}]^2\right]\\ =\mathrm{E}[\mathrm{X}^2]-2\mathrm{E}[\mathrm{X}]\,\mathrm{E}[\mathrm{X}]+\mathrm{E}[\mathrm{X}]^2\\ =\mathrm{E}[\mathrm{X}^2]-\mathrm{E}[\mathrm{X}]^2\end{array}\tag{16}$$

Put another way, the variance can be derived as

$$V(X)=\sigma^2=\mu_2-(\mu_1)^2=\tilde{\mu}_2\tag{17}$$

E. From these definitions, we see that for the uniform distribution

$$V(X)=\mu_2-(\mu_1)^2=\frac{1}{3}-\left(\frac{1}{2}\right)^2=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}\tag{18}$$

This can be verified directly by

$$\begin{array}{c}V(X)=\int_0^1\left(x-\frac{1}{2}\right)^2dx\\ =\int_0^1\left(x^2-x+\frac{1}{4}\right)dx\\ =\int_0^1 x^2dx-\int_0^1 xdx+\frac{1}{4}\int_0^1 dx\\ =\frac{1}{3}-\frac{1}{3}+\frac{1}{4}=\frac{4}{12}-\frac{6}{12}+\frac{3}{12}=\frac{1}{12}\end{array}\tag{19}$$