

Lecture XXXVII: Functions, Functionals and Solving the Nonlinear Pricing Model

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Functions and Functionals

- The standard starting point in algebra is the study of functions

$$f(x) \Rightarrow f : R^m \rightarrow R \quad (1)$$

so that $f(x)$ defines a mapping from a set of variables x into a resulting scalar space.

- Building on this core, we can define a functional as a map from a vector space (typically of functions) into a scalar space

$$x(t) = \{ x_1(t), x_2(t), \dots, x_m(t) \} \Rightarrow f(x) : R^m \rightarrow R \quad (2)$$

at the risk of oversimplification – a functional is a function of other functions.

- One application of this concept is in optimal control theory:

$$\max_{x(t), u(t)} \int_0^T F(t, x(t), u(t)) dt \quad (3)$$

$$\text{s.t. } x'(t) = g(t, x(t), u(t))$$

where $x(t)$ is the state variable and $u(t)$ is the control variable.

- The objective is to derive the functions $x(t)$ and $u(t)$ that satisfies three conditions:

$$\frac{\partial H}{\partial u} = 0 \Rightarrow f_u + \lambda g_u = 0$$

$$-\frac{\partial H}{\partial x} = \lambda'(t) \Rightarrow \lambda'(t) = -(f_x + \lambda g_x) \quad (4)$$

$$\frac{\partial H}{\partial \lambda} = x'(t) \Rightarrow x'(t) = g(t, x(t), u(t))$$

Simple Example of Optimal Control

- A simple example of the Optimal Control formulation

$$\begin{aligned} \max \int_0^1 (x + u) dt \\ \text{s.t. } x'(t) = 1 - u^2, x(0) = 1 \end{aligned} \quad (5)$$

The optimality conditions then become

$$H = x + u + \lambda (1 - u^2) \Rightarrow \begin{cases} \frac{\partial H}{\partial u} = 1 - 2\lambda u = 0 \\ -\frac{\partial H}{\partial x} = -1 = \lambda' \\ \frac{\partial H}{\partial \lambda} = 1 - u^2 = x'(t) \end{cases} \quad (6)$$

- Solving the Costate condition

$$-1 = \lambda'(t) \Rightarrow \lambda(t) = 1 - t. \quad (7)$$

- Substituting this result into the Optimality condition and solving

$$1 - 2\lambda u \Rightarrow 1 - 2(1 - t)u(t) = 0$$

$$\Rightarrow u(t) = \frac{1}{2(1 - t)} \quad (8)$$

Finally, using the Equation of Motion

$$x'(t) = 1 - \frac{1}{2(1 - t)^2}$$

$$x(t) = t - \frac{1}{4(1 - t)} + \frac{5}{4} \quad (9)$$

Hence, we used the information in the optimal control conditions to solve for a function (i.e., we solved for a function that maximized a functional).

Designing a Pricing Strategy

- Returning to the design of a pricing strategy, the objective function we want to maximize is

$$\int_T R(t) dt = \int_T [U(q(t), t) - W(t)] f(t) dt_1 \cdots dt_2. \quad (10)$$

Working backward $W(t)$ is defined by the pricing decision

$$W(t) \equiv \max_{q \in Q} U(q, t) - P(q) \quad (11)$$

$$R(t) \equiv P(q(t)) = U(q(t), t) - W(t)$$

the decision on pricing $P(q)$ defines the profit of the monopolist subject to the optimizing behavior of the consumer $U(q, t)$.

- Further, the Lagrange multiplier $\mu(t)$ is a form of the Costate variable $\lambda(t)$.
- The transformed maximization problem follows the general form of the Hamiltonian from Optimal Control

$$\int_T \{ [U(q(t), t) - W(t)] f(t) + [W_t(t) - U_t(q(t), t)] \times \mu(t) \} dt_1 \cdots dt_m \quad (12)$$

- This derivation leads to the conditions we developed in Lecture 36.

- **Allocative Optimality** of the assignment bundles:

$$U_q(q(t), t) f(t) - U_{qt}(q(t), t) \times \mu(t) \leq 0 \quad (13)$$

and this inequality is complementary to the feasibility constraint $q(t) \geq 0$.

$$\tilde{U}(q, t) \equiv U(q, t) - U_t(q, t) \times \frac{\mu(t)}{f(t)}. \quad (14)$$

- **Welfare Optimality** of the assignment of net benefits:

$$-f(t) - \sum_{i=1}^m \frac{\partial \mu_i(t)}{\partial t_i} \leq 0 \quad (15)$$

and this inequality is complementary to the participation constraint $W(t) \geq 0$ on the interior of the type domain (Euler condition).

- **Welfare Optimality on the Boundary:**

$$v(t) \times \mu(t) \leq 0 \quad (16)$$

if $t \in T$, and this inequality is complementary to the participation constraint $W(t) \geq 0$ (This is the transversality condition).

- “These conditions indicate that the key step in obtaining a solution is to construct the Lagrange multiplier μ from 15 and 16.”
 - “Once this multiplier has been obtained, the optimal assignment of types to bundles is obtained by solving the ordinary 13.”
 - “Moreover, at the bundle $q = q(t)$ the vector $p(q) \equiv P_q(q)$ of marginal prices must be $p(q) = U_q(q, t)$.”
 - “The tariff $P(q)$ is therefore obtained by integrating these marginal prices, using the participation constraint (where it binds) to determine the constant of integration.”
 - “The main computational task, therefore, is to construct the multiplier μ by solving the welfare-optimality condition 15 and the transversality condition 16, interpreted as equalities on T and ∂T .”

- **Integrability:** the vector field

$$U_t(q(t), t) \tag{17}$$

is integrable.

Example 1 – The Simple Case

- Consider a simple problem with the utility function

$$U(q, t) = \sum_{i=1}^m \left(t_i q_i - \frac{1}{2} q_i^2 \right) \quad (18)$$

- Taking the simple 3 good/3 attribute case

$$U(q, t) = \left(t_1 q_1 - \frac{1}{2} q_1^2 \right) + \left(t_2 q_2 - \frac{1}{2} q_2^2 \right) + \left(t_3 q_3 - \frac{1}{2} q_3^2 \right) \quad (19)$$

- The probability density function for these attributes is then defined as $f(t) = 1$ on the domain

$$T = \left\{ t \mid t_i \geq 0 \text{ and } \sum_i t_i^2 \leq r^2 \right\} \quad (20)$$

- One approach to solving differential equations is to “make a good guess” as to what the solution is and then work the conditions out. In this case, our guess on the solution is

$$W(t) = w(y(t)) \ni y(t) = \sum_i t_i^2 \quad (21)$$

- First from the **Allocative Optimality** condition

$$U_q(q(t), t) f(t) - U_{qt}(q(t), t) \times \mu(t) \leq 0 \Rightarrow \begin{cases} U_q(q, t) = t - 2q(t) \\ U_{qt}(q, t) \mu(t) = \mu(t) \\ f(t) = 1 \end{cases}$$

$$U_q(q(t), t) f(t) - U_{qt}(q(t), t) \times \mu(t) = t - 2q(t) - \mu(t)$$

$$\Rightarrow q(t) = t - \mu(t)$$

(22)

- From the **Incentive-Compatibility Constraint**:

$$W_t(t) = U_t(q(t), t) \Rightarrow \begin{cases} U_t(q(t), t) = q(t) \\ \Rightarrow W_t(t) = q(t) \end{cases} \quad (23)$$

- From our trial solution

$$w_t(t) = \frac{\partial w(t)}{\partial t} = w'(y(t))2t \quad (24)$$

- Thus from Equation 22

$$w'(y(t))2t = t - \mu(t) \quad (25)$$

- Next, we want to use the **Welfare Optimality** condition

$$-f(t) - \sum_{i=1}^m \frac{\partial \mu_i(t)}{\partial t_i} \leq 0 \quad (26)$$

- So we differentiate the result $w'(y(t))2t = t - \mu(t)$ to yield

$$w''(y(t)) \sum_i [2t_i]^2 + w'(y(t))2m = \sum_i \left[1 - \frac{\partial \mu_i(t)}{\partial t_i} \right] \quad (27)$$

Imposing $f(t) = 1$, the **Welfare Optimality** condition implies

$$\sum_i^m \frac{\partial \mu_i(t)}{\partial t_i} = -f(t) = -1 \quad (28)$$

Substituting this result into the right hand side of Equation 27 yields

$$w''(y(t)) \sum_i [2t_i]^2 + w'(y(t))2m = m + 1 \quad (29)$$

$$w''(y)4y + w'(y)2m = m + 1$$

- Not to get too deep in the weeds – The solution of Equation 29 subject to the boundary conditions is

$$w'(y) = \frac{1}{2} \left(1 + \frac{1}{m} \left[1 - \frac{r^m}{y^{m/2}} \right] \right) \quad (30)$$

- The values of the Lagrange multipliers or Costate conditions are then

$$\mu(t) = \frac{1}{m} \left[\left(\frac{r}{\sqrt{y(t)}} \right)^m - 1 \right] t \quad (31)$$

- The end of the story is that the optimal pricing structure becomes $p(q) = \mu(t(q))$ where $t(\cdot)$ is defined as the inverse map $t \rightarrow q(t) = t - \mu(t)$.

Computational Methods – Direct Optimization

- The concept is that exact solutions such as the one described above are rare.
- One alternative is to solve this problem using **Direct Optimization**.
- The set T of types is divided into a set of $\hat{T}$ discrete grid points with a “mesh” size of δ .

- The problem can then be solved by maximizing

$$\max_{q_t, W_t} \sum_{t \in \hat{T}} [U(q_t, t) - W_t] f_t \delta^m \quad (32)$$

$$\text{s.t. } W_{t+\delta e_i} - W_t = \frac{\partial U(q_t, t)}{t_i} \delta$$

where the constraint is a discretized form of the Incentive-Compatibility constraint.

- “Software for constrained optimization must ultimately solve an equivalent set of equations, but typically takes no account of the special structure of the problem – notably the simplification of the welfare-optimality condition via the Divergence Theorem.”