

Lecture XXXVI: Nonlinear Pricing and Mechanism Design

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1 Separating Equilibrium

Separating Equilibrium

- The simplest way to envision this problem is as a separating equilibrium:
 - Suppose that there are a large number of consumers whose demand for a good is a function of an group of attributes $T = \{ t_1, \dots t_m \}$.
 - The utility function for each consumer for a bundle of goods $q \in Q$ is a function of those attributes – $U(q, t)$.
 - These consumption goods are produced by a monopolist that chooses the price $P(q)$.
 - The monopolist knows the distribution of the attributes, but does not know the collection of attributes for any specific consumer.

- The goal of the monopolist is to design a pricing mechanism to maximize profit.
- The consumer's problem is to maximize utility

$$W(t) \equiv \max_{q \in Q} U(q, t) - P(q) \quad (1)$$

Making the additional assumption that we have expressed the utility function in terms of dollars (i.e., like the indirect utility function), $W(t)$ is the dollar value of the consumer's level of utility (think of as something like the willingness to pay).

- This dollar value is a function of the consumer's attributes.
- The consumer's utility function implies a "revenue function" for the monopolist

$$R(t) \equiv P(q(t)) = U(q(q), t) - W(t). \quad (2)$$

- Two necessary conditions

- **Incentive-Compatibility Constraint:**

$$W_t(t) = U_t(q(t), t) \rightarrow \left[\begin{array}{c} \frac{\partial U(q(t), t)}{\partial t_1} = \sum_{i=1}^I \frac{U}{\partial q_i} \frac{\partial q_i(t)}{\partial t_1} + \frac{\partial U}{\partial t_1} \\ \vdots \\ \frac{\partial U(q(t), t)}{\partial t_m} = \sum_{i=1}^I \frac{U}{\partial q_i} \frac{\partial q_i(t)}{\partial t_m} + \frac{\partial U}{\partial t_m} \end{array} \right] \quad (3)$$

if the bundle $q(t)$ is in the interior of Q . – The envelope of the consumer's decisions.

- **Participation Constraint** – $W(t) \geq 0$. – The consumer retains the right to forego purchasing.

- Assuming that the seller has no cost to simplify the maximization problem, the seller chooses two functions $q(t)$ and $W(t)$ to maximize

$$\int_T R(t) dt \equiv \int_T [U(q(t), t) - W(t)] f(t) dt_1 \cdots dt_m \quad (4)$$

subject to the incentive-compatibility and participation constraints.

$$\int_T \{ [U(q(t), t) - W(t)] f(t) + [W_t(t) - U_t(q(t), t)] \times \mu(t) \} dt_1 \cdots dt_m \quad (5)$$

- This problem can be solved using Euler's theorem applied to the calculus of variations and transversality condition.
- "To understand the source of these conditions, it is useful to eliminate the term involving the gradient W_t by using the form of multidimensional integration by parts obtained from the Divergence Theorem."
- This sounds bad. Think about the standard integration by parts

$$\begin{aligned}d(uv) &= u dv + v du \\ \Rightarrow u dv &= d(uv) - v du \\ \Rightarrow \int u dv &= \int d(uv) - \int v du\end{aligned}\tag{6}$$

- Transforming this into an integration by parts

$$\begin{aligned}\int_a^b u dv &= \int_a^b d(uv) - \int_a^b v du \\ \int_a^b u dv &= [u(b)v(b) - u(a)v(a)] - \int_a^b v du\end{aligned}\tag{7}$$

- From the original formulation

$$\int_T \{ [U(q(t), t) - W(t)] f(t) + [W_t(t) - U_t(q(t), t)] \times \mu(t) \} dt_1 \cdots dt_m$$

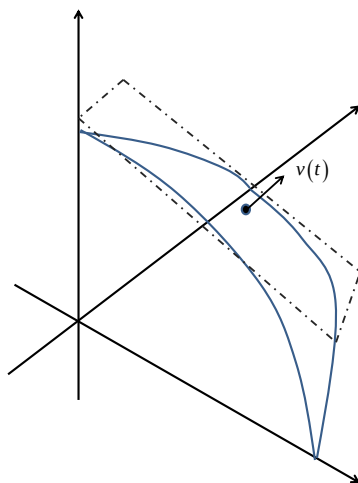
$$= \int_T \{ (U(q(t)) - W(t)) f(t) + W_t(t) \times \mu(t) - U_t(q(t), t) \times \mu(t) \} dt_1 \cdots dt_m \quad (8)$$

- Focusing on the $W_t(t) \times \mu(t)$ term

$$\int_T W_t(t) \times \mu(t) dt = \int_{\partial T} W(t) [v(t) \cdot \mu(t)] ds(t) - \int_T W(t) \sum_{i=1}^m \frac{\partial \mu_i(t)}{\partial t_i} dt \quad (9)$$

- “Here, $v(t)$ is the outward-pointing normal vector at a point t on the boundary ∂T of T , and $\int_{\partial T} ds(t_1) \cdots ds(t_m)$ represents the surface integral over the boundary; the summation is called the *divergence* of μ , often denoted $\nabla \mu$.”

Outward Pointing Normal Vector



Solution

- **Allocative Optimality** of the assignment bundles:

$$U_q(q(t), t) f(t) - U_{qt}(q(t), t) \times \mu(t) \leq 0 \quad (10)$$

and this inequality is complementary to the feasibility constraint $q(t) \geq 0$.

$$\tilde{U}(q, t) \equiv U(q, t) - U_t(q, t) \times \frac{\mu(t)}{f(t)}. \quad (11)$$

- **Welfare Optimality** of the assignment of net benefits:

$$-f(t) - \sum_{i=1}^m \frac{\partial \mu_i(t)}{\partial t_i} \leq 0 \quad (12)$$

and this inequality is complementary to the participation constraint $W(t) \geq 0$ on the interior of the type domain (Euler condition).

- **Welfare Optimality on the Boundary:**

$$v(t) \times \mu(t) \leq 0 \quad (13)$$

if $t \in T$, and this inequality is complementary to the participation constraint $W(t) \geq 0$ (This is the transversality condition).

- “These conditions indicate that the key step in obtaining a solution is to construct the Lagrange multiplier μ from 12 and 13.”
 - “Once this multiplier has been obtained, the optimal assignment of types to bundles is obtained by solving the ordinary 10.”
 - “Moreover, at the bundle $q = q(t)$ the vector $p(q) \equiv P_q(q)$ of marginal prices must be $p(q) = U_q(q, t)$.”
 - “The tariff $P(q)$ is therefore obtained by integrating these marginal prices, using the participation constraint (where it binds) to determine the constant of integration.”
 - “The main computational task, therefore, is to construct the multiplier μ by solving the welfare-optimality condition 12 and the transversality condition 13, interpreted as equalities on T and ∂T .”