

Lecture XXXV: Finding One Nash Equilibrium

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1 Setup

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Setup

- We start with a two person game with strategy sets $S = \{s_{i1}, \dots, s_{i_{m_i}}\}$ for player i , $i = 1, 2$.
- Let U_i be the payoff matrix for player i , with rows indexed by strategies for player i and columns indexed by the strategies for player $l \neq i$. Without loss of generality assume that all payoffs are positive.
- Thinking of the Prisoner's dilemma

$$U_1 = \begin{bmatrix} 1/2 & \text{Cooperate} & \text{Defect} \\ \text{Cooperate} & 12 & 1 \\ \text{Defect} & 10 & 3 \end{bmatrix} \quad (1)$$

$$U_2 = \begin{bmatrix} 2/1 & \text{Cooperate} & \text{Defect} \\ \text{Cooperate} & 9 & 2 \\ \text{Defect} & 8 & 4 \end{bmatrix}$$

- Thinking of an “initial strategy” for agent 1 of $p_1 = (0.75, 0.25)$ – the payoff for agent 1 is

$$p_{21} \times 0.75 \times 12 + p_{22} \times 0.25 \times 1 \quad (2)$$

with $p_{-1} = (p_{21}, p_{22})$

- The Nash equilibrium is a pair of column vectors $p_i \in R^m$ satisfying

$$U_i \times p_{-i} + r_i = v_i \times 1 \quad (3)$$

subject to $p_i \geq 0$, $r_i \geq 0$, $p_i \times 1 = 1$. Here v_i is a scalar representing player i 's payoff, and $r_i \in R^M$.

- Again for the first player

$$\begin{bmatrix} 12 & 1 \\ 10 & 3 \end{bmatrix} \times \begin{bmatrix} p_{21} \\ p_{22} \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = v_i \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (4)$$

- We normalize this problem by dividing through by v_i (as long as $v_i > 0$ yielding

$$U_i \times \tilde{p}_{-i} + \tilde{r}_i = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (5)$$

where $\tilde{p}_i \geq 0$, $\tilde{p}_i \times 1 \leq 1/v_{-i}$ and $\tilde{p}_i \tilde{r}_i = 0$.

- Putting both players together

$$U = \begin{bmatrix} 0 & U_1 \\ U_2 & 0 \end{bmatrix} \quad (6)$$

$$x = \begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \end{bmatrix}, y = \begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{bmatrix}, q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The equation can then be rewritten as

$$Uy + x = q \quad (7)$$

where $x \geq 0$, $y \geq 0$, and $x \times y = 0$. This is exactly a complementarity problem.

Complementary Slackness

- The Complementary Solution is related to the duality in linear programming.
- Consider the linear programming problem

$$\begin{aligned} \max z &= c'x \\ \text{s.t. } Ax &\leq b \end{aligned} \tag{8}$$

- We could set this up as a Lagrange multiplier problem

$$\max z = c'x + y'(b - Ax) \tag{9}$$

where y is the vector of Lagrange multipliers.

- We can transform this expression to yield

$$\min \tilde{z} = b'y + x'(c - A'y) \quad (10)$$

Equation 10 is the dual of Equation 9.

- Inside this primal/dual relationship lies the complementarity problem.
- From Equation 9 we are familiar with the condition that if constraint i is binding ($b_i - A_i \cdot x = 0$) then the resource can have a positive value ($y_i \geq 0$).
- If the constraint is not binding ($b_i - A_i \cdot x > 0$) then the resource has zero value ($y_i = 0$).
- However, the dual also provides the same results for primary outputs. Specifically, if paying each input its marginal value yields a negative return ($c_j - A'_{\cdot j} y < 0$) the output level is zero ($x_j = 0$).
- Similarly, if an activity has a positive value ($x_j > 0$), its dual constraint is equal to zero ($c_j - A'_{\cdot j} y = 0$).

- Building up the application

$$\begin{bmatrix} U & I_m \end{bmatrix} \times \begin{bmatrix} y \\ x \end{bmatrix} = q \quad (11)$$

- Writing this matrix out

$$\begin{bmatrix} 0 & 0 & 12 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 10 & 3 & 0 & 1 & 0 & 0 \\ 9 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 8 & 4 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{21} \\ p_{22} \\ r_{11} \\ r_{12} \\ r_{21} \\ r_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad (12)$$