

Lecture XXXIV: Digging into Numerical Game Theory

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1 Different Forms of the Game Theory Equilibrium

Fixed Point Theorem

- **Nash equilibrium as a fixed point function:** Define $y : \Delta \rightarrow \Delta$ (from a point on the p simplex to another point on the p simplex) with i, j^{th} component

$$y_{ij}(p) = \frac{p_{ij} + g_{ij}(p)}{1 + \sum_j g_{ij}(p)} \quad (1)$$

$$\begin{aligned} \exists: z_{ij}(p) &= x_{ij}(p) - u_i(p) \\ \text{and } g_{ij}(p) &= \text{Max}[z_{ij}(p), 0] \end{aligned}$$

Then $p^* \in \Delta$ is a Nash equilibrium if and only if it is a fixed point of y , in other words $p^* = y(p^*)$.

Non-Linear Complementarity Problem

- **Nash equilibrium as a solution to a non-linear complementarity problem.**
- As a starting point, I would like to go back and rebuild the concept of complementary slackness from mathematical programming.
- Think about a simple linear programming problem

$$\begin{aligned} \text{Max } z &= 10x_1 + 15x_2 + 3x_3 + x_4 \\ \text{s.t. } x_1 + x_2 + x_3 + x_4 &\leq 10 \\ 15x_1 + 7x_2 + x_3 + x_4 &\leq 25 \end{aligned} \tag{2}$$

- The GAMS results for this problem is presented in Table 1

Table: Linear Programming Results

Variable	Level	λ
x_1	0.00	-21.00
x_2	2.50	0.00
x_3	7.50	0.00
x_4	0.00	-2.00

- In fact, the two constraints are “binding” which means that they have a marginal value. Rewriting the problem slightly

$$\begin{aligned} y_1 &= 10 - x_1 - x_2 - x_3 - x_4 = 0 \ni x_1 = 2.50 \text{ and } x_2 = 7.50 \\ y_2 &= 25 - 15x_1 - 7x_2 - x_3 - x_4 = 0 \ni x_1 = 2.50 \text{ and } x_2 = 7.50 \end{aligned} \quad (3)$$

- For the first constraint $\tilde{\lambda}_1 = 1.00$ and for the second constraint $\tilde{\lambda}_2 = 2.00$.

- We could write the solution of this problem as a system of complementary slackness or complementarity problems

$$\begin{bmatrix} y_1 = 10 - x_1 - x_2 - x_3 - x_4 \\ y_2 = 25 - 15x_1 - 7x_2 - x_3 - x_4 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' \begin{bmatrix} \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = 0 \quad (4)$$

- The function $z : P \rightarrow R^m$ as developed in our discussion satisfies $p_i \cdot z_i(p) = 0$ for all p and each i (i.e., p_i and $z_i(p)$ are orthogonal).
- A point $p^* \in \Delta$ is a Nash equilibrium if and only if $z(p^*) \leq 0$.
- The Nonlinear Complementarity Problem (NCP) on Δ then consists of finding a point $p \in \Delta$ with $z(p) \leq 0$.
- Such a point p is complementary to $z(p)$ (i.e., $p_{ij} \cdot z_{ij}(p) = 0$ for all i, j).

Stationary Point Problem

- **Nash equilibrium as a stationary point problem.** A point $p^* \in \Delta$ is a Nash equilibrium if and only if it satisfies

$$(p_i - p_i^*) \cdot x_i(p^*) \leq 0 \quad (5)$$

for all i and $p \in \Delta$.

Minimization on a Polytope

- **Nash equilibrium as a minimum of a function on a polytope.**
- The word polytope is used to mean a number of related, but slightly different mathematical objects. A convex polytope may be defined as the convex hull of a finite set of points (which are always bounded), or as a bounded intersection of a finite set of half-spaces (*Wolfram Mathworld* <http://mathworld.wolfram.com/Polytope.html>).
- Starting with the concept of a half-space, graphically, I could draw a line in a cartesian plane and divide all space into a half space. Figure 1 presents the “first cut” for a polytope.

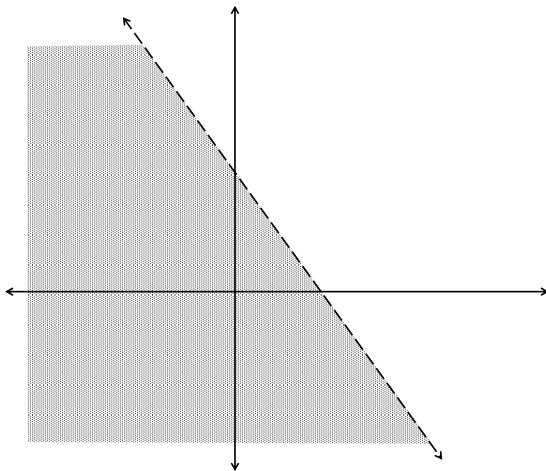


Figure: Simple Half-Space

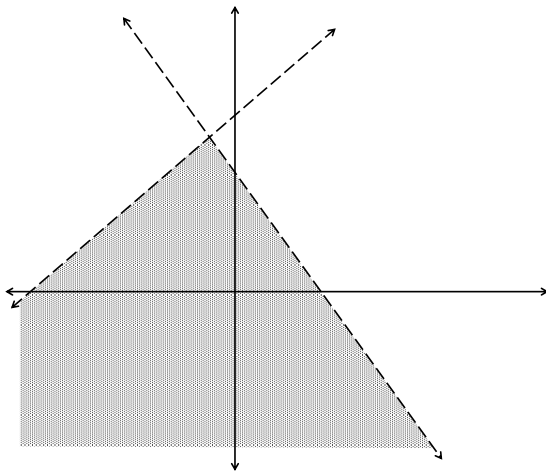


Figure: Slightly More Complicated Polytope

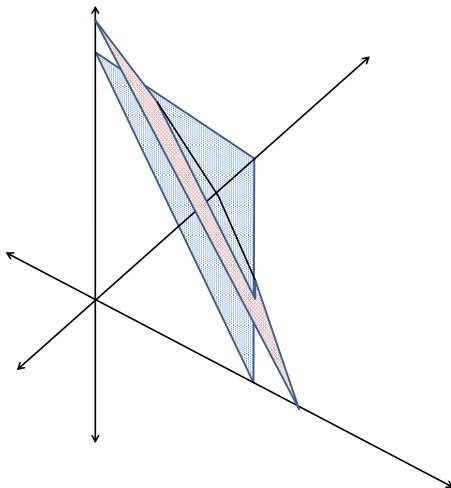


Figure: Two Planes in 3-Dimension Space

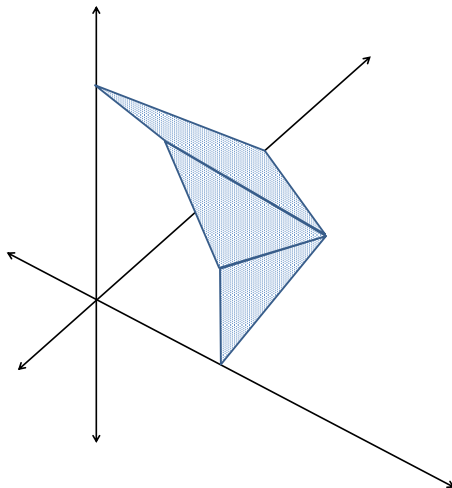


Figure: Three Plane Polytope in 3-Dimensions

- Define a real valued function $v : \Delta \rightarrow R$ by

$$v(p) = \sum_{i \in N} \sum_{1 \leq j \leq m_i} [g_{ij}(p)]^2 \quad (6)$$

This is a continuous real valued function satisfying $v(p) \geq 0$ for all p .

- Further, p^* is a Nash equilibrium if and only if it is a global minimum of v . In other words $v(p) = 0$.