

Lecture XXXIII: An Introduction to Finite Games

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November 15, 2012

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Setting up the Game

- Consider a n - person (finite) game.
 - Assume there are a set of $N = \{ 1, \dots n \}$ players and assume that each player has a set of $S_i = \{ s_{i1}, \dots s_{im_i} \}$ strategies mixed strategies (so player i has m_i possible mixed strategies).
 - For each $i \in N$, assume that we can define a payoff function $u_i : S \rightarrow R$ where $S = \prod_{i \in N} S_i$.

- In the Adena Springs Ranch example, we could look at three sets of players - { the ranch owner ($R - 1$), the water management district ($W - 2$), and environmental groups ($E - 3$) }
 - The ranch owner could have three strategies: retain its current operations (C), go for the full amount (F), or some modification (i.e., agree not to pump in some states of nature - M) - $S_1 = \{ C, F, M \}$.
 - The water management district could have three strategies: approve the permit (A), deny the permit (D), or impose some restrictions (i.e., modify the maximum flows in certain states - R) - $S_2 = \{ A, D, R \}$.
 - Finally, the environmental groups could have two possible strategies: protest (P) and not protest (NP) - $S_3 = \{ P, NP \}$
- In this case $S = \prod_{i \in N} S_i$ includes the 18 cases presented in Table 5

Strategy Combinations

$\{ C, A, P \}$	$\{ F, A, P \}$	$\{ M, A, P \}$
$\{ C, D, P \}$	$\{ F, D, P \}$	$\{ M, D, P \}$
$\{ C, R, P \}$	$\{ F, R, P \}$	$\{ M, R, P \}$
$\{ C, A, NP \}$	$\{ F, A, NP \}$	$\{ M, A, NP \}$
$\{ C, D, NP \}$	$\{ F, D, NP \}$	$\{ M, D, NP \}$
$\{ C, R, NP \}$	$\{ F, R, NP \}$	$\{ M, R, NP \}$

- In each case we can work out the gains to the producers and the environmental groups. It is somewhat more difficult to derive the value to the water management district.

Setting up the Solution

- The basic nature of the solution process is to define a set of $0 \leq p_{ij} \leq 1$ (similar to probabilities) that weight each of the strategies.
- Given the assumptions we can derive a value for each cell

$$\{C, A, P\} \Rightarrow \{s_{1C}, s_{2A}, s_{3P}\} \Rightarrow \begin{cases} u_1(s_{1C}, s_{2A}, s_{3P}) \\ u_2(s_{1C}, s_{2A}, s_{3P}) \\ u_3(s_{1C}, s_{2A}, s_{3P}) \end{cases} \quad (1)$$

- For simplicity, we could envision the strategy combinations in Table 5 as "pure strategies." Implicitly, we could define p_1 as the pure strategy of the ranch owner to maintain his current operation

$$p_1 \Rightarrow \begin{Bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} p_1(s_{11}) \\ p_1(s_{12}) \\ p_1(s_{13}) \end{Bmatrix} \quad (2)$$

- So the strategy is valid as long as the elements sum to one.

$$p_1 \Rightarrow \begin{Bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 0.25 \\ 0.25 \end{Bmatrix} \quad (3)$$

From the Chapter

- Let P_i be the set of real valued functions on S_i .
- For elements $p_i \in P_i$ we use the notation $p_{ij} = p_i(s_{ij})$.
- Let $P = \prod_{i \in N} P_i$ and let $m = \sum_{i \in N} m_i$.
 - In the current example
 $m = 8 = m_1 (= 3) + m_2 (= 3) + m_3 (= 2)$.

- Then P is isomorphic to R^M . (from Mathworld – Formally, an isomorphism is a bijective morphism. Informally, an isomorphism is a map that preserves sets and relationships among elements.)

$$x = \ln(z) \Rightarrow \text{if } z_1 > z_2 \Rightarrow x_1 > x_2 \quad (4)$$

- We denote points in P by $p = (p_1, \dots, p_n)$, where $p_i = (p_{i1}, \dots, p_{im_i})$.
- If $p \in P$ and $p'_i \in P_i$, we use the notation (p'_i, p_{-i}) for the element $q \in P$ satisfying $q_i = p'_i$ and $q_j = p_j$ for $i \neq j$ (Basically a way of excluding one strategy).

- The payoff function u is extended to have the domain R^M by the rule

$$u_i(p) = \sum_{s \in S} p(s) u_i(s) \quad (5)$$

- To add some detail, assume that we are looking at the combination of strategies

$$p = (p_1, p_2, p_3) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.50 \\ 0.25 \\ 0.25 \end{bmatrix}, \begin{bmatrix} 0.33 \\ 0.33 \\ 0.33 \end{bmatrix} \right\} \quad (6)$$

- The value of the outcome for the ranch manager is then

$$u_i(p) = \sum_i p_{2i} \left[\sum_j p_{3j} u_1(s_{11}, s_{2i}, s_{3j}) \right]$$

$$\begin{aligned} &= 0.50 [0.33u_1(s_{11}, s_{21}, s_{31}) + 0.33u_1(s_{11}, s_{21}, s_{32}) + 0.33u_1(s_{11}, s_{21}, s_{33})] + \\ &0.25 [0.33u_1(s_{11}, s_{22}, s_{31}) + 0.33u_1(s_{11}, s_{22}, s_{32}) + 0.33u_1(s_{11}, s_{22}, s_{33})] + \\ &0.25 [0.33u_1(s_{11}, s_{23}, s_{31}) + 0.33u_1(s_{11}, s_{23}, s_{32}) + 0.33u_1(s_{11}, s_{23}, s_{33})] \end{aligned} \quad (7)$$

- Let $\Delta_i = \left\{ p_i \in P_i : \sum_j p_{ij} = 1, p_i \geq 0 \right\}$ be the set of probability measures on s_i

$$\Delta = \prod_{i \in N} \Delta_i \subseteq R_M \quad (8)$$

(Remembering our discussions from General Equilibrium – this is basically a simplex).

- **Definition 1:** We say that $p^* \in P$ is a *Nash equilibrium* if $p^* \in \Delta$ and for all $i \in N$ and all $p_i \in \Delta_i$, $u_i(p_i, p_{-i}^*) \leq u_i(p^*)$.

- It is easier to see what is going on if we define some more functions:
 - $x_{ij} = u_i(s_{ij}, p_{-i})$ { The utility of the set of all strategies with the current strategy removed }.
 - $z_{ij}(p) = x_{ij}(p) - u_i(p)$ { The value of the prospective strategy over the index strategy }.
 - $g_{ij} = \text{Max}[z_{ij}(p), 0]$.
- The Nash equilibrium is then a "fixed point" correspondence. Define the best response correspondence as $\lambda : \Delta \rightarrow \Delta$

$$\lambda(p) = \text{argmax}_{q \in \Delta} \left[\sum_{i \in N} u_i(q_i, p_{-i}) \right] \quad (9)$$

The $p^* \in \Delta$ is a Nash equilibrium if and only if it is a fixed point of λ , in other words $p^* \in \lambda(p^*)$.

- To find the Nash equilibrium, we define $y : \Delta \rightarrow \Delta$ with components i, j as

$$y_{ij}(p) = \frac{p_{ij} + g_{ij}(p)}{1 + \sum_j g_{ij}(p)} \quad (10)$$