

Lecture XXIX: CES Demand and Construction of the CGE Model

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- 1 CES Demand Functions
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- 3 Computing the General Equilibrium Solution

CES Demand Function

- In Lecture 29, we derived the input demand curves for the Constant Elasticity of Substitution specification. Now if we assume that land, labor, and capital are applied to produce agricultural and manufactured goods, we can specify a production system as depicted in Table 1

Table: CES Coefficients

$\alpha_{c,f}$ (c - Commodity, f - Factor)	Land (α_{c1})	Labor (α_{c2})	Capital (α_{c3})	σ_c
Y_a - Agriculture	0.70	0.10	0.20	0.95
Y_m - Manufacturing	0.05	0.45	0.50	1.05

- Given these parameters, we have two production functions

$$\begin{aligned}
 Y_a &= [0.6870x_1^{-0.0526} + 0.0886x_2^{-0.0526} + 0.1838x_3^{-0.0526}]^{-19.0} \\
 Y_m &= [0.0577x_1^{0.0476} + 0.4675x_2^{0.0476} + 0.5167x_3^{0.0476}]^{21.0}
 \end{aligned}
 \tag{1}$$

- The input demands conditional on the output level can be defined based on these parameters

$$\begin{aligned}
 x_{1a} &= \frac{0.70Y_a}{w_1^{0.95} [0.70w_1^{0.05} + 0.10w_2^{0.05} + 0.20w_3^{0.05}]^{-19.0}} \\
 x_{2a} &= \frac{0.10Y_a}{w_2^{0.95} [0.70w_1^{0.05} + 0.10w_2^{0.05} + 0.20w_3^{0.05}]^{-19.0}} \\
 x_{3a} &= \frac{0.20Y_a}{w_3^{0.95} [0.70w_1^{0.05} + 0.10w_2^{0.05} + 0.20w_3^{0.05}]^{-19.0}}
 \end{aligned}
 \tag{2}$$

$$\begin{aligned}
 x_{1m} &= \frac{0.05 Y_m}{w_1^{1.05} \left[0.05 w_1^{-0.05} + 0.45 w_2^{-0.05} + 0.05 w_3^{-0.05} \right]^{21.0}} \\
 x_{2m} &= \frac{0.45 Y_m}{w_2^{1.05} \left[0.05 w_1^{-0.05} + 0.45 w_2^{-0.05} + 0.05 w_3^{-0.05} \right]^{21.0}} \\
 x_{3m} &= \frac{0.50 Y_m}{w_3^{1.05} \left[0.05 w_1^{-0.05} + 0.45 w_2^{-0.05} + 0.05 w_3^{-0.05} \right]^{21.0}}
 \end{aligned} \tag{3}$$

- It is obvious that given output levels and input prices, the demand equations define the quantity of input demanded.
 - If we assume that each factor is given some initial endowment, then the excess demand for each input can be defined as

$$\xi_i(w, Y_a, Y_b) = x_{ia}(w, Y_a) + x_{im}(w, Y_m) - E_i \quad (4)$$

- Taking the example a step further, we could write the expression of in an "object form" (like Mathematica) as

$$\xi_i(w, p, I) = x_{ia}(w, Y_a[p, I]) + x_{im}(w, Y_m[p, I]) - E_i \quad (5)$$

where $Y_a[p, I]$ is the consumer demand for agricultural output as a function of the output prices p and level of consumer income I and for a similar construction for the consumer demand for manufacturing output.

- Thus, to complete the picture we have to formulate the consumer's demand equations.
- If we again assume the CES form (this time for the utility function) we can derive these demand relationships as

$$\begin{aligned} \max_{Y_a, Y_m} & \left[\beta_a^{\frac{1}{\phi}} Y_a^{\frac{\phi-1}{\phi}} + \beta_m^{\frac{1}{\phi}} Y_m^{\frac{\phi-1}{\phi}} \right]^{\frac{\phi}{\phi-1}} \\ \text{s.t. } & Y_a p_a + Y_m p_m = I \end{aligned} \quad (6)$$

- Forming the Lagrangian problem (repeating the derivations from Lecture 29).

$$\begin{aligned}
 L &= \left[\beta_a^{\frac{1}{\phi}} Y_a^{\frac{\phi-1}{\phi}} + \beta_m^{\frac{1}{\phi}} Y_m^{\frac{\phi-1}{\phi}} \right]^{\frac{\phi}{\phi-1}} - \mu [I - Y_a p_a - Y_m p_m] \\
 \frac{\partial L}{\partial Y_a} &= \frac{\phi-1}{\phi} \beta_a^{\frac{1}{\phi}} Y_a^{\frac{\phi-1}{\phi}-1} \frac{\phi}{\phi-1} \left[\beta_a^{\frac{1}{\phi}} Y_a^{\frac{\phi-1}{\phi}} + \beta_m^{\frac{1}{\phi}} Y_m^{\frac{\phi-1}{\phi}} \right]^{\frac{\phi}{\phi-1}-1} - \mu p_a = 0 \\
 &\Rightarrow \beta_a^{\frac{1}{\phi}} Y_a^{-\frac{1}{\phi}} \left[\beta_a^{\frac{1}{\phi}} Y_a^{\frac{\phi-1}{\phi}} + \beta_m^{\frac{1}{\phi}} Y_m^{\frac{\phi-1}{\phi}} \right]^{\frac{1}{\phi-1}} - \mu p_a = 0 \\
 \frac{\partial L}{\partial Y_m} &= \beta_m^{\frac{1}{\phi}} Y_m^{-\frac{1}{\phi}} \left[\beta_a^{\frac{1}{\phi}} Y_a^{\frac{\phi-1}{\phi}} + \beta_m^{\frac{1}{\phi}} Y_m^{\frac{\phi-1}{\phi}} \right]^{\frac{1}{\phi-1}} - \mu p_m = 0 \\
 \frac{\partial L}{\partial \mu} &= I - Y_a p_a - Y_m p_m = 0
 \end{aligned}
 \tag{7}$$

- Again taking the ratio of the first and second first-order conditions yields

$$\frac{\beta_a^{\frac{1}{\phi}} Y_a^{-\frac{1}{\phi}} \left[\beta_a^{\frac{1}{\phi}} Y_a^{\frac{\phi-1}{\phi}} + \beta_m^{\frac{1}{\phi}} Y_m^{\frac{\phi-1}{\phi}} \right]^{\frac{1}{\phi-1}}}{\beta_m^{\frac{1}{\phi}} Y_M^{-\frac{1}{\phi}} \left[\beta_a^{\frac{1}{\phi}} Y_a^{\frac{\phi-1}{\phi}} + \beta_m^{\frac{1}{\phi}} Y_m^{\frac{\phi-1}{\phi}} \right]^{\frac{1}{\phi-1}}} = \frac{\mu p_a}{\mu p_M}$$

$$\left[\frac{\beta_a}{\beta_m} \frac{Y_m}{Y_a} \right]^{\frac{1}{\phi}} = \frac{p_a}{p_m} \quad (8)$$

$$\frac{\beta_m}{\beta_a} \frac{Y_m}{Y_a} = \left[\frac{p_a}{p_m} \right]^{\phi}$$

$$Y_m = \left[\frac{p_a}{p_m} \right]^{\phi} \frac{\beta_m}{\beta_a} Y_a$$

- Substituting this solution into the first-order condition with respect to the Lagrange multiplier yields

$$I - Y_a p_a - \left(\left[\frac{p_a}{p_m} \right]^\phi \frac{\beta_m}{\beta_a} Y_a \right) p_m = 0$$

$$I - Y_a p_a - p_m^{1-\phi} p_a^\phi \frac{\beta_m}{\beta_a} Y_a = 0$$

$$I = \left[p_a + p_m^{1-\phi} p_a^\phi \frac{\beta_m}{\beta_a} \right] Y_a \quad (9)$$

$$I \beta_a = \left[p_a \beta_a + p_m^{1-\phi} p_a^\phi \beta_m \right] Y_a$$

$$\frac{I \beta_a}{\left[p_a \beta_a + p_m^{1-\phi} p_a^\phi \beta_m \right]} = Y_a$$

- To reconcile this result with the previous results in Lecture 29, we factor p_a^ϕ out of the denominator yielding

$$Y_a = \frac{I\beta_a}{p_a^\phi \left[p_a^{1-\phi}\beta_a + p_m^{1-\phi}\beta_m \right]} \text{ or}$$

$$Y_i = \frac{I\beta_i}{p_i^\phi \left[\sum_j p_j^{1-\phi}\beta_j \right]} \quad (10)$$

- For our purposes we assume that $\beta_a = 0.35$, $\beta_m = 0.65$, and $\phi = 0.75$ so the demand for each good becomes

$$Y_a = \frac{0.35I}{p_a^{0.75} [0.35p_a^{-0.25} + 0.65p_m^{-0.25}]} \quad (11)$$

$$Y_m = \frac{0.65I}{p_m^{0.75} [0.35p_a^{-0.25} + 0.65p_m^{-0.25}]}$$

Construction of the CGE Model

- It is now possible to specify the excess demand for an output price/ income combination

$$\xi_i(p, I) = \frac{I\beta_i}{p_i^\phi \left[\sum_j p_j^{1-\phi} \beta_j \right]} - \left[\sum_k \alpha_{ki}^{\frac{1}{\sigma}} x_{ki}(w, Y_k)^{\frac{\sigma_i-1}{\sigma_i}} \right]^{\frac{\sigma_i}{\sigma_i-1}} \quad (12)$$

- We close the model by specifying income as the return to owned factors

$$I = w_1 [x_{1a}(w, Y_a) + x_{1m}(w, Y_m)] + w_2 [x_{2a}(w, Y_a) + x_{2m}(w, Y_m)] + w_3 [x_{3a}(w, Y_a) + x_{3m}(w, Y_m)] \quad (13)$$

- The procedure is then that the demand equations based on a set of prices generate a set of desired outputs.
- These desired outputs at a given set of input prices generate input demands.
- Computing a general equilibrium then involves finding a set of prices where the excess demands for all outputs and inputs is equal to zero.
- An additional assumption is that no profits are retained in production

$$\pi = p_a Y_a + p_m Y_m - w_1 (x_{11} + x_{21}) - w_2 (x_{12} + x_{22}) - w_3 (x_{13} + x_{23}) = 0 \quad (14)$$

Computing the General Equilibrium Solution

- There are two procedures for computing the general equilibrium values.
- Scarf's algorithm – this algorithm develops simplexes within feasible space.
- A Newton-Type algorithm – based on the optimization models we have developed.

Scarf's Algorithm

- Specifically, given k current solutions, it determines the solution to be dropped and a better point to be added.
- This procedure is guaranteed to find an optimal, but is slow and tedious.

Newton's Algorithm

- Technically, this procedure is not guaranteed to find an optimum, but if the supply and demand equations are well behaved typically does. In addition, it is easier to program.
- The Newtons method algorithms are actually variants of the Gauss-Siedel procedure.
- Assume that you have a vector valued function based on x , $G(x)$.
- The first order Taylor series expansion of this function can be written as:

$$G(x) = G(x_0) + \nabla_x G(x_0)(x - x_0) \quad (15)$$

- Note that by assumption, $G(x)$ is of the same order as x .

- Thus, solving this expression for $G(x) = 0$ implies

$$\begin{aligned} x - x_0 &= -(\nabla_x G(x_0))^{-1} G(x_0) \\ x &= x_0 - (\nabla_x G(x_0))^{-1} G(x_0) \end{aligned} \quad (16)$$

- Given that the supply and demand relationships are well behaved, the second term on the right hand side will converge to zero as $G(x_0)$ converges to zero.
- Applying this approach to the excess demand expression

$$p = p_0 - (\nabla_p \xi(p_0, I))^{-1} \xi(p_0, I) \quad (17)$$