

Lecture XXIX: Fixed Points and CES Functions

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1 Fixed Points and General Equilibrium

- Brouwer's Fixed Point Theorem
- Kakutani's Theorem
- Gale-Nikaido

2 Constant Elasticity of Substitution

Fixed Points and General Equilibrium

- In Lecture 28, we discussed an intuitive formulation of the "Fixed Point" formulation. Mathematically

$$\begin{aligned} p &\leftrightarrow p[\xi(p)] \\ \xi &\leftrightarrow \xi[p(\xi)] \end{aligned} \tag{1}$$

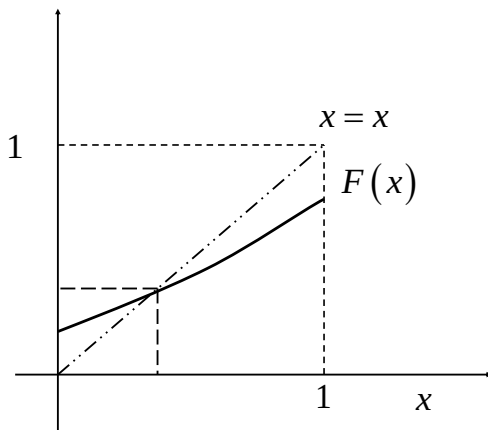
- The Fixed Point theorems show that under certain conditions p can be chosen such that $p \rightarrow \xi(p) \rightarrow p$, or that the mappings is "fixed."
- Brouwer's Fixed Point Theorem: A continuous function from a ball (of any dimension) to itself must leave at least one point fixed – Armstrong, M.A. 1983. *Basic Topology*. New York: Springer-Verlag, p. 110.

Brouwer's Fixed Point Theorem

- Proof for dimension 1:
 - Assume that a function is defined on a unit interval $I \in [0, 1]$, $f : I \rightarrow I$ (i.e., the function maps into itself).
 - Further assume that the function is continuous.
 - There must be a point $x \in I$ such that $f(x) = x$.
 - The proof is by contradiction.

- Both $x, f(x) \in I$, starting from $x = 0$ and moving to the right.
- If $x = 0 = f(x)$, 0 is the fixed point.
- Otherwise, $f(x) > x$.
- Moving to the right, if $f(x)$ is continuous then either $f(x) = x$ for some x or $f(x) > x$ for some x or $f(x) > x$.
- If $f(x) > x$ for all points $x \in [0, 1)$ then either $f(1) = 1$ or $f(x)$ violates the domain.

Brouwer's Fixed Point Theorem



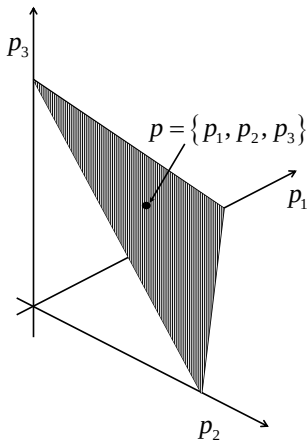
Kakutani's Theorem

- This fixed point theorem can be extended to 2 or more dimensions, but these proofs require axioms beyond this brief introduction.
- Kakutani's Theorem: Let the Point-To-Set mapping $X \rightarrow \phi(X)$ of the simplex S into itself be upper semicontinuous. Assume that for each X , $\phi(X)$ is a nonempty, closed and convex set. Then there is a fixed point $X \in \phi(X)$.

The Simplex

- In general, a simplex is a hyperplane spanned by k points in a k dimensional space.
 - For example, in E^2 space two points span a line.
 - In E^3 space, three points span a plane.
 - In general, $k + 1$ points determine a convex set called the simplex of dimension k .
- In the current analysis, we are interested in $k + 1$ prices.
 - A given set of these $k + 1$ prices span a k^{th} dimension hyperplane.
 - For three prices, we span a two dimensional plane in E^3 space.

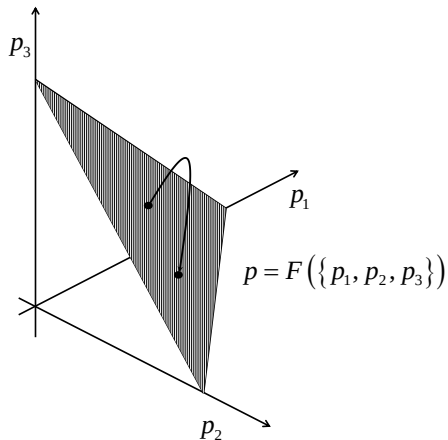
Price Simplex



Gale-Nikaido

- The functional maps then involve $f : R_+^{k-1} \rightarrow R_+^{k-1}$. This mapping is some mathematical operation that takes as its input one point on the plane and gives another point on the plane.
- The Gale-Nikaido mapping can be used as a functional mapping of the simplex back onto itself.

Gale-Nikaido Mapping



- Using our previous definitions of excess supply and demand, the Gale Nikaido mapping is defined as

$$p_i = \frac{p_i + \max [0, \xi_i (p)]}{1 + \sum_{i=1}^N \max [0, \xi_i (p)]} \quad (2)$$

If the point obeys Brouwer's fixed point theorem, we have

$$p_i^* = \frac{p_i^* + \max [0, \xi_i (p^*)]}{1 + \sum_{i=1}^N \max [0, \xi_i (p^*)]} \quad (3)$$

- To demonstrate the equilibrium, we must show that the denominator of the right hand side is equal to one, or that

$$\sum_{i=1}^N \max [0, \xi_i (p)] = 0 \Leftrightarrow p_i^* = \frac{p_i^* + 0}{1 + 0} \quad (4)$$

To show this, let

$$\begin{aligned} c = 1 + \sum_{i=1}^N \max [0, \xi_i (0)] &\Rightarrow cp_i^* = p_i^* + \max [0, \xi_i (p^*)] \\ &\Rightarrow (c - 1) p_i^* = \max [0, \xi_i (p^*)]. \end{aligned} \quad (5)$$

Mathematically c cannot be less than one due to the $\max [0, x]$ on the right hand side.

- The economic context of this expression then depends on ruling out $c > 0$. If $c > 0$, since $p_i^* > 0$ there must be an excess demand $\xi_i(p^*) > 0$ which would violate the complementary slackness condition in Walras law

$$\sum_{i=1}^N p_i \xi_i(p) = 0. \quad (6)$$

Constant Elasticity of Substitution

- The choice of input levels that minimize the cost of production for any set of input prices and a fixed level of production can be expressed as

$$\begin{aligned} \min_{x_i} \sum_i x_i w_i \\ \text{s.t. } Y = f(x). \end{aligned} \quad (7)$$

- Limiting for exposition purposes the set of inputs to 3 and focusing on the constant elasticity of substitution (CES) production function yields

$$\begin{aligned} \min_{x_1, x_2, x_3} w_1 x_1 + w_2 x_2 + w_3 x_3 \\ \text{s.t. } Y = \left[\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}} + \alpha_3^{\frac{1}{\sigma}} x_3^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \end{aligned} \quad (8)$$

- Forming the Lagrangian for the optimization problem

$$L = w_1 x_1 + w_2 x_2 + w_3 x_3 + \lambda \left(Y - \left[\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}} + \alpha_3^{\frac{1}{\sigma}} x_3^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \right) \quad (9)$$

- The first-order conditions for this minimization problem are then

$$\begin{aligned} \frac{\partial L}{\partial x_1} = w_1 - \lambda \alpha_1^{\frac{1}{\sigma}} \frac{\sigma-1}{\sigma} x_1^{\frac{\sigma-1}{\sigma}-1} \frac{\sigma}{\sigma-1} \\ \times \left[\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}} + \alpha_3^{\frac{1}{\sigma}} x_3^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}-1} = 0. \end{aligned} \quad (10)$$

- Given that

$$\frac{\sigma - 1}{\sigma} - 1 = \frac{\sigma - 1}{\sigma} - \frac{\sigma}{\sigma} = -\frac{1}{\sigma} \quad (11)$$

and

$$\frac{\sigma}{\sigma - 1} - 1 = \frac{\sigma}{\sigma - 1} - \frac{\sigma - 1}{\sigma - 1} = \frac{1}{\sigma - 1} \quad (12)$$

The first-order conditions become

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= w_1 - \lambda \alpha_1^{\frac{1}{\sigma}} x_1^{-\frac{1}{\sigma}} \left[\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}} + \alpha_3^{\frac{1}{\sigma}} x_3^{\frac{\sigma-1}{\sigma}} \right]^{\frac{1}{\sigma-1}} = 0 \\ \frac{\partial L}{\partial x_2} &= w_2 - \lambda \alpha_2^{\frac{1}{\sigma}} x_2^{-\frac{1}{\sigma}} \left[\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}} + \alpha_3^{\frac{1}{\sigma}} x_3^{\frac{\sigma-1}{\sigma}} \right]^{\frac{1}{\sigma-1}} = 0 \\ \frac{\partial L}{\partial x_3} &= w_3 - \lambda \alpha_3^{\frac{1}{\sigma}} x_3^{-\frac{1}{\sigma}} \left[\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}} + \alpha_3^{\frac{1}{\sigma}} x_3^{\frac{\sigma-1}{\sigma}} \right]^{\frac{1}{\sigma-1}} = 0 \\ Y - \left[\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}} + \alpha_3^{\frac{1}{\sigma}} x_3^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}-1} &= 0 \end{aligned} \quad (13)$$

- The ratio of the first-order condition for the first input to the first-order condition for the second input yields

$$\frac{w_1}{w_2} = \frac{\lambda \alpha_1^{\frac{1}{\sigma}} x_1^{-\frac{1}{\sigma}} \left[\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}} + \alpha_3^{\frac{1}{\sigma}} x_3^{\frac{\sigma-1}{\sigma}} \right]^{\frac{1}{\sigma-1}}}{\lambda \alpha_2^{\frac{1}{\sigma}} x_2^{-\frac{1}{\sigma}} \left[\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}} + \alpha_3^{\frac{1}{\sigma}} x_3^{\frac{\sigma-1}{\sigma}} \right]^{\frac{1}{\sigma-1}}}$$

$$\frac{w_1}{w_2} = \left[\frac{\alpha_1 x_2}{\alpha_2 x_1} \right]^{\frac{1}{\sigma}} \quad (14)$$

$$\left[\frac{w_1}{w_2} \right]^{\frac{1}{\sigma}} = \frac{\alpha_1 x_2}{\alpha_2 x_1} \Rightarrow x_2 = \left[\frac{w_1}{w_2} \right]^{\frac{1}{\sigma}} \frac{\alpha_2}{\alpha_1} x_1$$

- Simply replacing the price and constant for x_3 yields

$$x_3 = \left[\frac{w_1}{w_3} \right]^{\frac{1}{\sigma}} \frac{\alpha_3}{\alpha_1} x_1 \quad (15)$$

- Substituting these results into the first-order condition with respect to the Lagrange multiplier yields

$$\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} \left(\left\{ \frac{w_1}{w_2} \right\}^{\sigma} \frac{\alpha_2}{\alpha_1} x_1 \right)^{\frac{\sigma-1}{\sigma}} + \alpha_3^{\frac{1}{\sigma}} \left(\left\{ \frac{w_1}{w_3} \right\}^{\sigma} \frac{\alpha_3}{\alpha_1} x_1 \right)^{\frac{\sigma-1}{\sigma}} \Bigg] \frac{\sigma-1}{\sigma-1} = 0. \quad (16)$$

- Focusing on the term inside the brackets

$$\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \frac{\alpha_2^{\frac{1}{\sigma} + \frac{\sigma-1}{\sigma}}}{\alpha_1^{\frac{\sigma-1}{\sigma}}} \left(\frac{w_1}{w_2} \right)^{\sigma-1} x_1^{\frac{\sigma-1}{\sigma}} + \frac{\alpha_3^{\frac{1}{\sigma} + \frac{\sigma-1}{\sigma}}}{\alpha_1^{\frac{\sigma-1}{\sigma}}} \left(\frac{w_1}{w_3} \right)^{\sigma-1} x_1^{\frac{\sigma-1}{\sigma}}. \quad (17)$$

- Next, we solve

$$\begin{aligned}\alpha_1^{\frac{1}{\sigma}} &= \frac{\alpha_1}{\alpha_1^z} \\ \alpha_1^{\frac{1}{\sigma}} &= \alpha_1^{1-z} \\ \frac{1}{\sigma} &= 1 - z \\ z &= 1 - \frac{1}{\sigma} = \frac{\sigma - 1}{\sigma}\end{aligned}\tag{18}$$

substituting this solution into the first term of Equation 17

$$\frac{\alpha_1}{\alpha_1^{\frac{\sigma-1}{\sigma}}} x_1^{\frac{\sigma-1}{\sigma}} + \frac{\alpha_2^{\frac{1}{\sigma} + \frac{\sigma-1}{\sigma}}}{\alpha_1^{\frac{\sigma-1}{\sigma}}} \left(\frac{w_1}{w_2} \right)^{\sigma-1} x_1^{\frac{\sigma-1}{\sigma}} + \frac{\alpha_3^{\frac{1}{\sigma} + \frac{\sigma-1}{\sigma}}}{\alpha_1^{\frac{\sigma-1}{\sigma}}} \left(\frac{w_1}{w_3} \right)^{\sigma-1} x_1^{\frac{\sigma-1}{\sigma}}\tag{19}$$

- Finally, we multiply the first term by a form of one

$$\left(\frac{w_1}{w_1}\right)^{\sigma-1} \quad (20)$$

this yields

$$\left(\frac{w_1}{w_1}\right)^{\sigma-1} x_1^{\frac{\sigma-1}{\sigma}} + \frac{\alpha_2^{\frac{1}{\sigma} + \frac{\sigma-1}{\sigma}}}{\alpha_1^{\frac{\sigma-1}{\sigma}}} \left(\frac{w_1}{w_2}\right)^{\sigma-1} x_1^{\frac{\sigma-1}{\sigma}} + \frac{\alpha_3^{\frac{1}{\sigma} + \frac{\sigma-1}{\sigma}}}{\alpha_1^{\frac{\sigma-1}{\sigma}}} \left(\frac{w_1}{w_3}\right)^{\sigma-1} x_1^{\frac{\sigma-1}{\sigma}}. \quad (21)$$

- Factoring out the terms involving $\alpha_1^{\frac{\sigma-1}{\sigma}}$, w_1 , and x_1 yields

$$\alpha_1^{\frac{1-\sigma}{\sigma}} w_1^{\sigma-1} x_1^{\frac{\sigma-1}{\sigma}} \left[\sum_{i=1}^3 \alpha_i w_i^{1-\sigma} \right] \quad (22)$$

- The first-order condition from the Lagrange multiplier then becomes

$$Y - \left[\alpha_1^{\frac{1-\sigma}{\sigma}} w_1^{\sigma-1} x_1^{\frac{\sigma-1}{\sigma}} \left(\sum_{i=1}^3 \alpha_i w_i^{1-\sigma} \right) \right]^{\frac{\sigma}{\sigma-1}} = 0$$

$$Y = \alpha_1^{-1} w_1^{\sigma} x_1 \left[\sum_{i=1}^3 \alpha_i w_i^{1-\sigma} \right]^{\frac{\sigma}{\sigma-1}}$$

$$x_i = \frac{Y \alpha_1}{w_1^{\sigma} \left[\sum_{i=1}^3 \alpha_i w_i^{1-\sigma} \right]^{\frac{\sigma}{\sigma-1}}} \quad (23)$$