

Lecture XVII: Unconstrained to Constrained Algorithm

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1 Unconstrained Algorithm

2 Linear Equality Constrained Problem

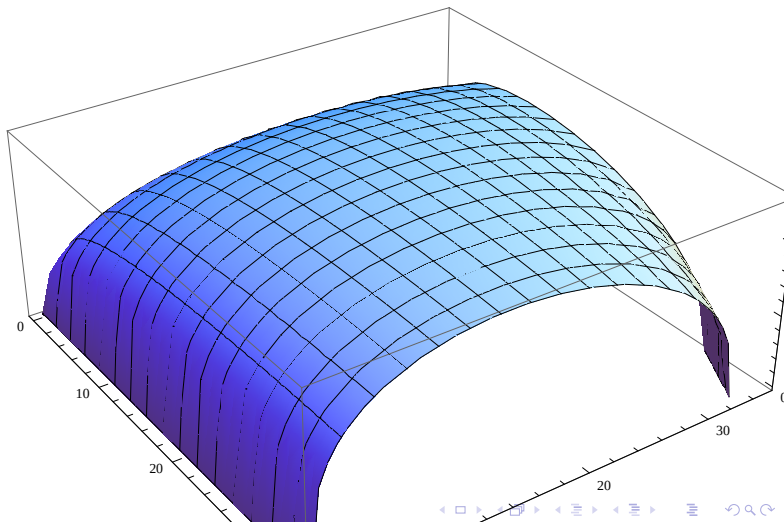
Let's start with a linearly constrained problem

$$\begin{aligned} U(x_1, x_2, x_3) &= x_1^{0.2} x_2^{0.3} x_3^{0.4} \\ \text{s.t. } x_1 + 2x_2 + 3x_3 &\leq 100. \end{aligned} \tag{1}$$

Folding the constraint into the objective function

$$x_1^{0.2} x_2^{0.3} \left(\frac{100 - x_1 - 2x_2}{3} \right)^{0.4} \tag{2}$$

Constrained Imposed Unconstrained Problem



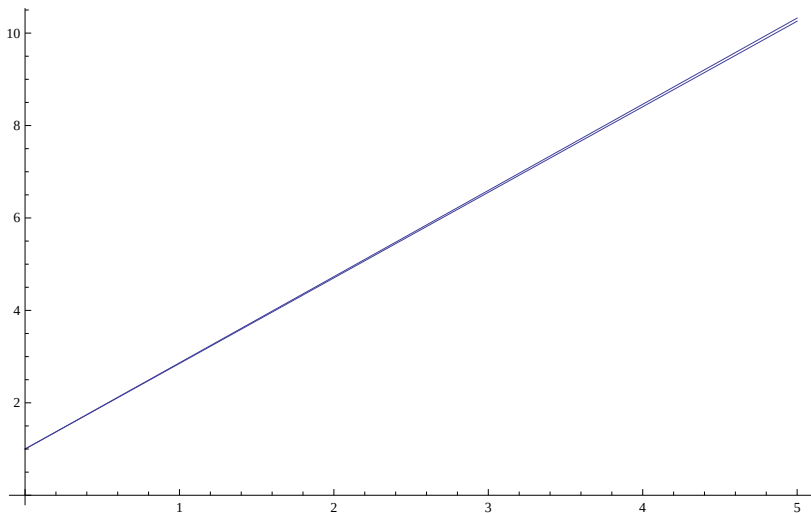
Consider starting at point $x_0 = \begin{pmatrix} 1 & 1 \end{pmatrix}$. The Newton step is then

$$x_1 = \begin{pmatrix} 1.0 & 1.0 \end{pmatrix} - \alpha_0 \begin{pmatrix} 0.7868 & 1.1719 \end{pmatrix} \begin{bmatrix} -0.6494 & 0.2292 \\ 0.2292 & -0.8678 \end{bmatrix}$$

$$x_1 = \begin{pmatrix} 1.0 & 1.0 \end{pmatrix} - \alpha_0 \begin{pmatrix} 1.8650 & 1.8516 \end{pmatrix} \quad (3)$$

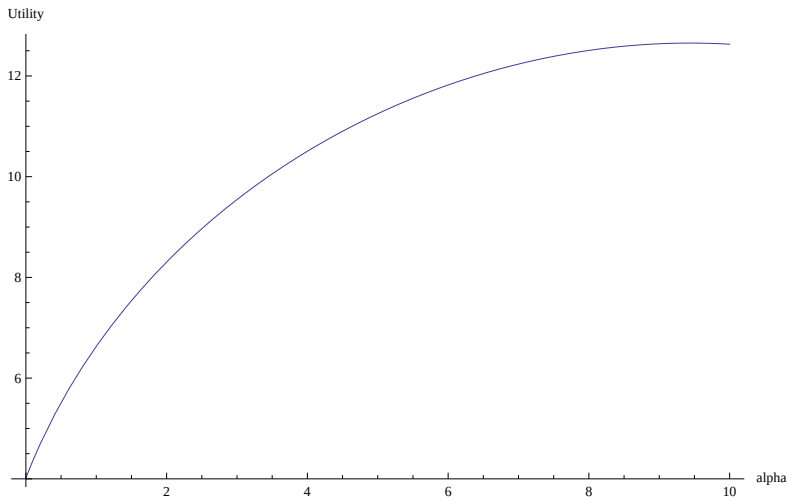
Consider the projection in two dimensions presented in Figure 6.

Projection toward Optimality



Graphing the objective function down that slice from Figure 6 yields the graph in Figure 8.

Slice down the optimal projection



Looking at this graph, it is obvious that $\alpha_0 = 1.0$ is too short. For the time being, let us assume that $\alpha_0 = 8.25$. Comparing the two results

$$\begin{aligned} x_1 &= \begin{pmatrix} 1.0 & 1.0 \end{pmatrix} - 1.0 \begin{pmatrix} 1.8650 & 1.8516 \end{pmatrix} \\ \nabla_x U(x_1, x_2) &= \begin{pmatrix} 0.4338 & 0.6395 \end{pmatrix} \\ x_1^* &= \begin{pmatrix} 1.0 & 1.0 \end{pmatrix} - 8.25 \begin{pmatrix} 1.8650 & 1.8516 \end{pmatrix} \\ \nabla_x U(x_1, x_2) &= \begin{pmatrix} 0.0549 & 0.0347 \end{pmatrix} \end{aligned} \tag{4}$$

Linear Equality Constrained Problem

Notice that I have been imposing the constraint

$$x_3 = \frac{100 - x_1 - 2x_2}{3} \quad (5)$$

for the budget constraint. At point $x = (1.0 \ 1.0)$ this implies that $x_3 = 97/3$. Going back to the original problem, we assume that the initial solution is $x_0 = (1 \ 1 \ 97/3)$. The A matrix is

$$A = (1 \ 2 \ 3) \Rightarrow Z = \begin{pmatrix} -3 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} \quad (6)$$

Following on the Newton-Raphson method, we have

$$\tilde{x}_1 = \tilde{x}_0 - \nabla_x f(x) Z' (Z \nabla_{xx}^2 f(x) Z')^{-1} \quad (7)$$

However, let's rewrite this slightly

$$\begin{aligned} \tilde{x}_1 &= \tilde{x}_0 + \alpha p_1 \\ p_1 &= -\nabla_x f(x) Z' (Z \nabla_{xx}^2 f(x) Z')^{-1} \end{aligned} \quad (8)$$

Why the difference? Consider the null space formulation again.
We know that

$$x_1 = x_0 + Z'p \Rightarrow x_1 = b \Leftrightarrow A(x_0 + Z'p) \quad (9)$$

Thus, we are going to rewrite the step equation as

$$\begin{aligned} \tilde{x}_1 &= \tilde{x}_0 + \alpha p_1 Z \\ p_1 &= -\nabla_x f(x) Z' (Z \nabla_{xx}^2 f(x) Z')^{-1} \end{aligned} \quad (10)$$

In this case

$$p_1 = (-1.85604 \quad 1.85155) \quad (11)$$

so the new $\tilde{x}$ becomes

$$\begin{aligned} \tilde{x}_1 &= (1 \quad 1 \quad 97/3) - (-1.85604 \quad 1.85155) \begin{bmatrix} -3 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} \\ \tilde{x}_1 &= (2.86504 \quad 2.85155 \quad 30.4773) \end{aligned} \quad (12)$$