

Lecture XV: Necessary and Sufficient Conditions under Nonlinear Constraints

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1 Simple Constraint

2 Optimality Conditions

- Unconstrained Multivariate [UP]
- Linearly Constrained Multivariate [LEP]
- Linearly Constrained Inequality [LIP]
- Nonlinearly Constrained [NIP]

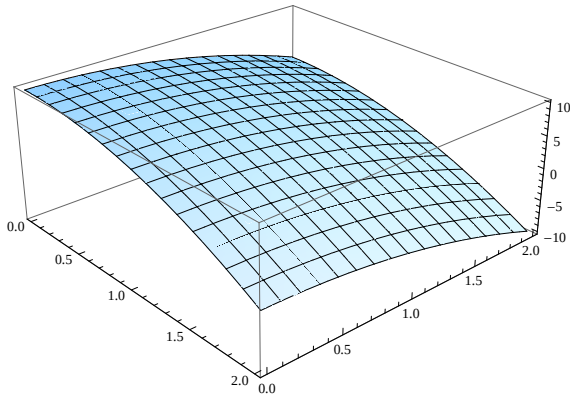
Simple Constraint

- To begin our discussion, let us assume that we want to maximize a quadratic utility function subject to an elliptical constraint.
- Abstracting away from the objective function I want to develop the meaning of the first order Taylor expansion of this elliptical constraint at a given point.
- Let the elliptical constraint be

$$g(x) = 10 - 3x_1^2 - 2x_2^2 \quad (1)$$

- Graphically, Equation 1 is a cone "mound" of sorts as depicted in Figure 4.

Nonlinear Ellipse

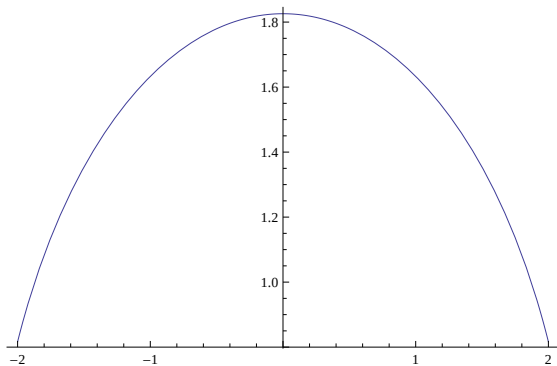


- If we solve equation 1 for x_1 in terms of x_2 we get

$$x_1 = -\frac{\sqrt{2(5-x_2^2)}}{\sqrt{3}} \quad x_1 = \frac{\sqrt{2(5-x_2^2)}}{\sqrt{3}} \quad (2)$$

- Graphing this result in two dimensions (or the implicit function of x_1 in terms of x_2) yields Figure 6.

Ellipse in Two Dimensions



- Next, we want to linearize this function at a particular point, say $x_2 = .75$. Solving the implicit function at this point yields $x_1 = 1.71998$.
- The Taylor series expansion of this constraint is then defined as

$$G(x) \approx G(x^*) + \nabla_x G(x^*) dx \quad (3)$$

- The Jacobian of this expansion defined as

$$\nabla_x G(x) = \begin{bmatrix} \frac{\partial G_1(x)}{\partial x_1} & \frac{\partial G_1(x)}{\partial x_2} & \dots & \frac{\partial G_1(x)}{\partial x_n} \\ \frac{\partial G_2(x)}{\partial x_1} & \frac{\partial G_2(x)}{\partial x_2} & \dots & \frac{\partial G_2(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial G_m(x)}{\partial x_1} & \frac{\partial G_m(x)}{\partial x_2} & \dots & \frac{\partial G_m(x)}{\partial x_n} \end{bmatrix} \quad (4)$$

is the cornerstone of the linear approximation. For the current problem, the Jacobian at the point of approximation is

$$\nabla_x G(x) = \begin{bmatrix} -5x_1 & -4x_2 \end{bmatrix} = \begin{bmatrix} -10.3199 & -3.000 \end{bmatrix}. \quad (5)$$

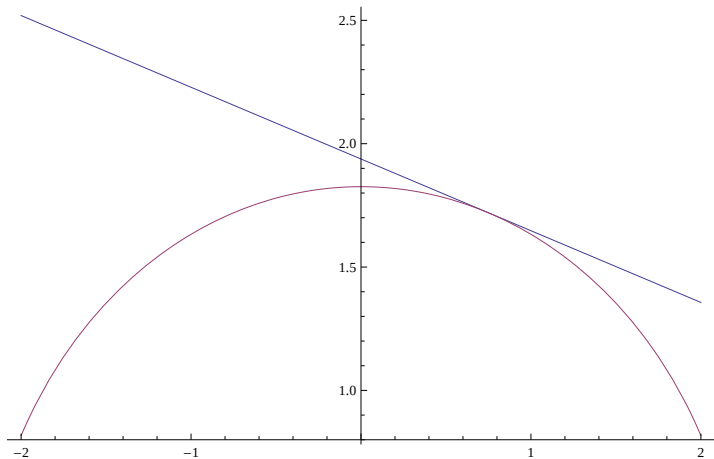
- The linear relationship around that point then becomes

$$G(x) - G(x^*) = \begin{bmatrix} -10.3199 & -3.000 \end{bmatrix} \begin{bmatrix} x_1 - 1.71998 \\ x_2 - 0.75 \end{bmatrix} \quad (6)$$

$$= 20 - 10.3199x_1 - 3.00x_2$$

- Graphically, this expression is presented in Figure 10

Linear Approximation of Constraint



- Consider the maximization problem

$$\begin{aligned} \max_{x_1, x_2} & 5.0x_1^{0.35}x_2^{0.45} \\ \text{s.t. } & g(x_1, x_2) = 0 \end{aligned} \quad (7)$$

evaluated at the point $x^* = (1.2076 \ 1.67706)$. The gradient of the objective function at this point is

$$\nabla_x f(x) = [1.95359 \ 1.80865]. \quad (8)$$

- The Jacobian becomes

$$\nabla_x g(x) = \begin{bmatrix} -6x_1 \\ -4x_2 \end{bmatrix} = \begin{bmatrix} -7.24564 \\ -6.70824 \end{bmatrix} \quad (9)$$

- The NullSpace for this matrix is

$$Z(x) = \begin{bmatrix} -0.67937 \\ 0.73380 \end{bmatrix} \quad (10)$$

- The optimality conditions are then

$$\begin{bmatrix} 1.95359 & 1.80865 \end{bmatrix} \begin{bmatrix} -0.679371 \\ 0.73380 \end{bmatrix} = -0.00003349 \quad (11)$$

- Note that

$$\nabla_{xx} g_1(x) = \begin{bmatrix} -6.0 & 0.0 \\ 0.0 & -4.0 \end{bmatrix} \quad (12)$$

thus as long as $\lambda_1 \geq 0$, the second order conditions are met.

Unconstrained Multivariate [UP]

$$\|\nabla_x f(x)\| = 0 \quad (13)$$

$\nabla_{xx}^2 f(x)$ is positive semidefinite

Linearly Constrained Multivariate [LEP]

$$Ax = b$$

$$||Z' \nabla_x f(x)|| = 0 \text{ or } \nabla_x f(x) = A' \lambda \quad (14)$$

$Z' \nabla_x f(x) Z$ is positive semidefinite

Linearly Constrained Inequality [LIP]

$$Ax \geq b \text{ and } \hat{A}x = \hat{b}$$

$$||Z' \nabla_x f(x)|| = 0 \text{ or } \nabla_x f(x) = \hat{A}' \lambda \quad (15)$$

$\lambda \geq 0$ for the active constraints

$Z' \nabla_{xx}^2 f(x) Z$ is positive definite

Nonlinearly Constrained [NIP]

$$G(x) \geq 0$$

$$Z(x)' \nabla_x f(x) = 0 \text{ or } \nabla_x f(x) = A(x) \lambda$$

$$Z(x)' W(x, \lambda) Z(x) \text{ is positive semidefinite} \quad (16)$$

$$W(x, \lambda) = \nabla_{xx}^2 f(x) - \sum_{i=1}^m \lambda_i \nabla_{xx}^2 G_i(x)$$