

Lecture XIII: Linear Constraints – Gradient and Lagrange

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- 1 Review of Linear Constraints
 - Kuhn-Tucker Conditions

Review of Linear Constraints

- As previously developed, the first order necessary conditions for a linearly constrained equality can be written as

$$\nabla_x f(x) Z = 0 \quad (1)$$

for the previous example

$$\begin{aligned} \max_{x_1, x_2, x_3, x_4, x_5} \quad & u(x) = x_1^{0.15} x_2^{0.20} x_3^{0.03} x_4^{0.13} x_5^{0.23} \\ \text{s.t.} \quad & 5x_1 + 2x_2 + 3x_3 + 4x_4 + x_5 \leq 100. \end{aligned} \quad (2)$$

- The potential optimum point was

$$x^* = \left\{ \begin{array}{ccccc} 4.0541 & 13.5135 & 1.3514 & 4.3919 & 31.0811 \end{array} \right\} .$$

The gradient at the point of optimum is

$$\begin{aligned} \nabla_x u(x) &= \left[\begin{array}{ccccc} 0.15 \frac{u(x)}{x_1} & 0.20 \frac{u(x)}{x_2} & 0.03 \frac{u(x)}{x_3} & 0.13 \frac{u(x)}{x_4} & 0.23 \frac{u(x)}{x_5} \end{array} \right] \\ &\quad \left[\begin{array}{ccccc} 0.2072 & 0.0829 & 0.1243 & 0.1657 & 0.0414 \end{array} \right] \end{aligned} \quad (3)$$

given that $u(x) = 5.5986$.

- Thus, given the NullSpace matrix the optimality condition becomes

$$\nabla_x f(x) Z = \begin{bmatrix} 0.2072 & 0.0829 & 0.1243 & 0.1657 & 0.0414 \end{bmatrix} \\ \times \begin{bmatrix} -1.0 & -4.0 & -3.0 & -2.0 \\ 0.0 & 0.0 & 0.0 & 5.0 \\ 0.0 & 0.0 & 5.0 & 0.0 \\ 0.0 & 5.0 & 0.0 & 0.0 \\ 5.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \quad (4)$$

Kuhn-Tucker Conditions

- An alternative specification for this constraint (i.e., like the Kuhn-Tucker formulation) becomes

$$\nabla_x f(x) = A' \lambda \quad (5)$$

- where λ are a vector of Lagrange multipliers.
 - Another way to look at the problem is that the Lagrange multipliers are that set of constants which equate the gradient with the linear constraints.
 - To offer an intuitive proof

$$\begin{aligned} L &= f(x) + \lambda_1 (b_1 - a_1 x) + \lambda_2 (b_2 - a_2 x) \\ \nabla_x L &= \nabla_x f(x) - \lambda_1 a_1 - \lambda_2 a_2 = 0 \\ \Rightarrow \nabla_x f(x) &= \lambda_1 a_1 + \lambda_2 a_2 \\ &= A' \lambda \end{aligned} \quad (6)$$

The general concept of the Lagrange multiplier and the null space matrix then work together.

- Another concept that I want to discuss is the fact that a equality constrained problem can have Lagrange multipliers of either side.
 - Intuitively, the Lagrange multiplier is the change in the objective function associated with a one unit increase in the right hand side of the constraint.
 - Therefore, in a maximization problem:
 - If the constraint cuts the frontier below the global maximum (assuming that the objective function resembles a quadratic function), the Lagrange multiplier will take on a positive value implying that an increase in the right hand side of the constraint will increase the objective function value.
 - Similarly, if the constraint cuts the frontier above the global maximum (again assuming that the objective function resembles a quadratic function), the Lagrange multiplier will take on a negative value implying that an increase in the right hand side of the constraint will cause the objective function to decline.

- Building on our previous discussion

$$\nabla_x u(x)' = \begin{bmatrix} 0.2072 \\ 0.0829 \\ 0.1243 \\ 0.1657 \\ 0.0414 \end{bmatrix} = \begin{bmatrix} 5.0 \\ 2.0 \\ 3.0 \\ 4.0 \\ 1.0 \end{bmatrix} \lambda \quad (7)$$

In this case, the solution is relatively straightforward. Divide the gradient by its last element

$$\frac{\nabla_x u(x)}{0.04143} = [5 \quad 2 \quad 3 \quad 4 \quad 1] \quad (8)$$

Thus, $\lambda = 24.1371$.

- Consider adding a "dummy constraint" to the formulation – It makes no economic sense. Let this constraint be that

$$x_1 + x_2 + x_3 + x_4 + x_5 = 5.0 \quad (9)$$

The new optimum point becomes

$x^* = \{ 1.014 \quad 1.351 \quad 0.203 \quad 0.878 \quad 1.554 \}$. The Lagrange formulation for this problem becomes

$$\nabla_x u(x) = \begin{bmatrix} 0.16331 \\ 0.16343 \\ 0.1631 \\ 0.1635 \\ 0.16339 \end{bmatrix} = \begin{bmatrix} 5.0 & 1.0 \\ 2.0 & 1.0 \\ 3.0 & 1.0 \\ 4.0 & 1.0 \\ 1.0 & 1.0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \quad (10)$$

- Solve

$$\begin{bmatrix} 0.16331 & 5.0 & 1.0 \\ 0.16343 & 2.0 & 1.0 \\ 0.1631 & 3.0 & 1.0 \\ 0.1635 & 4.0 & 1.0 \\ 0.16339 & 1.0 & 1.0 \end{bmatrix} \quad (11)$$