

Lecture XI: Projected Nonlinear Constraint and Linear Constraints

Charles B. Moss ¹

¹University of Florida

September 16, 2012

1 Linear Approximation

2 Linear Constraints

Linear Approximation

- As a starting point, imagine solving the problem

$$\min_{x_1, x_2, x_3, x_4, x_5} 10.00y_1 + 15.00y_2$$

$$\text{s.t. } y_1 f_1(x_1, x_2, x_3, x_4, x_5) = 0.15 \ln(x_1) + 0.25 \ln(x_2) + 0.05 \ln(x_3) + 0.10 \ln(x_4) + 0.45 \ln(x_5) \quad (1)$$

$$y_2 = f_2(x_1, x_2, x_3, x_4, x_5) = 0.30 \ln(x_1) + 0.15 \ln(x_2) + 0.25 \ln(x_3) + 0.25 \ln(x_4) + 0.05 \ln(x_5)$$

at $x_1 = 40.0$, $x_2 = 63.33$, $x_3 = 34.00$, $x_4 = 19.00$, and $x_5 = 105.00$. Random number – Right. The result of this value is $f_1(x) = 4.15548$ and $f_2(x) = 3.57932$.

- As a first step, let's define the Jacobian

$$\nabla_x \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} \frac{0.15}{x_1} & \frac{0.25}{x_2} & \frac{0.05}{x_3} & \frac{0.10}{x_4} & \frac{0.45}{x_5} \\ \frac{0.30}{x_1} & \frac{0.15}{x_2} & \frac{0.25}{x_3} & \frac{0.25}{x_4} & \frac{0.05}{x_5} \end{bmatrix} \quad (2)$$

At the point of interest

$$\nabla_x \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} 0.00375 & 0.003947 & 0.001471 & 0.005263 & 0.004286 \\ 0.0075 & 0.002368 & 0.007353 & 0.01316 & 0.0004762 \end{bmatrix} \quad (3)$$

- Letting

$$G_1 = \begin{bmatrix} 0.003750 & 0.003947 \\ 0.0075 & 0.002368 \end{bmatrix} \quad (4)$$

and

$$G_2 = \begin{bmatrix} 0.001471 & 0.005263 & 0.004286 \\ 0.0073529 & 0.01316 & 0.0004762 \end{bmatrix}. \quad (5)$$

- We can define the change around the point of the interest as

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = G_1^{-1} G_2 \begin{pmatrix} dx_3 \\ dx_4 \\ dx_5 \end{pmatrix} \quad (6)$$
$$= \begin{bmatrix} 1.23249 & 1.90476 & -0.399093 \\ -0.79832 & -0.47619 & 1.46485 \end{bmatrix} \begin{pmatrix} dx_3 \\ dx_4 \\ dx_5 \end{pmatrix}.$$

- Letting $dx_3 = 0.10$, $dx_4 = -0.10$, and $dx_5 = 0.05$ the resulting change is

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{bmatrix} -0.08719 \\ 0.04102 \end{bmatrix} \quad (7)$$

The new level is then $x_1 = 39.9128$, $x_2 = 63.3744$, $x_3 = 34.1$, $x_4 = 18.9$, and $x_5 = 105.05$. Comparing the results

Table: Constraint Values

Function	Original Value	New Value
$f_1(x)$	4.15548	4.15515
$f_2(x)$	3.57932	3.57820

Linear Constraints

- The general programming problem can be written

$$\begin{array}{ll} \max_x & f(x) \\ \text{s.t.} & G(x) = b \end{array} \quad (8)$$

can be restructured into the linearly constrained equality problem

$$\begin{array}{ll} \max_x & f(x) \\ \text{s.t.} & Ax = b \end{array} \quad (9)$$

- Mathematically one such problem is the portfolio choice problem where an individual minimizes the variance of an investment portfolio subject to an income constraint

$$\begin{aligned}\min_x f(x) &= x' \Omega x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} \\ \omega_{21} & \omega_{22} & \omega_{23} & \omega_{24} \\ \omega_{31} & \omega_{32} & \omega_{33} & \omega_{34} \\ \omega_{41} & \omega_{42} & \omega_{43} & \omega_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ &= \omega_{11}x_1^2 + 2\omega_{12}x_1x_2 + 2\omega_{13}x_1x_3 + 2\omega_{14}x_1x_4 + \omega_{22}x_2^2 + \\ &\quad 2\omega_{23}x_2x_3 + 2\omega_{24}x_2x_4 + \omega_{33}x_3^2 + 2\omega_{34}x_3x_4 + \omega_{44}x_4^2 \\ &\text{s.t. } \mu_1x_1 + \mu_2x_2 + \mu_3x_3 + \mu_4x_4 = Y^*\end{aligned}\tag{10}$$

Mathematically, solving this system is based on the null-space of the income constraint. In this case, setting the level of x_2 through x_4 determines the level of x_1

$$x_1 = \frac{Y^*}{\mu_1} - \frac{\mu_2}{\mu_1}x_2 - \frac{\mu_3}{\mu_1}x_3 - \frac{\mu_4}{\mu_1}. \quad (11)$$

- We could substitute for this equality into the original objective function

$$\begin{aligned} f^*(x) = & \omega_{11} \left(\frac{Y^*}{\mu_1} - \frac{\mu_2}{\mu_1} x_2 - \frac{\mu_3}{\mu_1} x_3 - \frac{\mu_4}{\mu_1} \right) + \\ & 2\omega_{12} x_2 \left(\frac{Y^*}{\mu_1} - \frac{\mu_2}{\mu_1} x_2 - \frac{\mu_3}{\mu_1} x_3 - \frac{\mu_4}{\mu_1} \right) + \\ & 2\omega_{13} x_3 \left(\frac{Y^*}{\mu_1} - \frac{\mu_2}{\mu_1} x_2 - \frac{\mu_3}{\mu_1} x_3 - \frac{\mu_4}{\mu_1} \right) + \\ & 2\omega_{14} x_4 \left(\frac{Y^*}{\mu_1} - \frac{\mu_2}{\mu_1} x_2 - \frac{\mu_3}{\mu_1} x_3 - \frac{\mu_4}{\mu_1} \right) + \\ & \omega_{22} x_2^2 + 2\omega_{23} x_2 x_3 + 2\omega_{24} x_2 x_4 + \omega_{33} x_3^2 + 2\omega_{34} x_3 x_4 + \omega_{44} x_4^2. \end{aligned} \quad (12)$$

reducing the multivariate constrained problem to a multivariate unconstrained problem. However, this scenario becomes impractical as the number of equations and variables grow.

- Another way to conceive the problem is by the formal definition of the null space.
- What we want to do is to develop a formalized way of making the statement that $p = (\bar{x} - \hat{x})$ implies that $A\bar{x} = b$ and $A\hat{x} = b$.
- Thus, $A\bar{x} = A\hat{x}$ implies that $A(\bar{x} - \hat{x}) = 0$ or that $Ap = 0$.
- The collection of all such vectors is referred to as the null space. Specifically, let the columns of matrix Z form a basis; then $AZ = 0$ defines the null space.

- In this case, we need to modify our concept of the neighborhood.
- Previously, we were interested in any projection around x^* , thus $x = x^* + p$.
- Now, we are only interested in those points that maintain feasibility.
- To accomplish this, we begin with a projection (or perturbation) in the same dimension as the null space.
- This perturbation is then mapped into the general space using the null space basis.

- Thus, the set of feasible x s becomes $x = x^* + Zp$.
- The second order expansion of the function then becomes

$$f(x) = f(x^*) + \nabla_x f(x^*) Zp + \frac{1}{2} p' Z' \nabla_{xx}^2 f(x) Zp. \quad (13)$$

- Again using the mechanics for a maximum, we have that

$$f(x) - f(x^*) = \nabla_x f(x^*) Zp + \frac{1}{2} p' Z' \nabla_{xx}^2 f(x^*) Zp \leq 0 \quad (14)$$

for any p . Hence, we derive the first necessary condition for a maximum that

$$\nabla_x f(x^*) Z = 0 \quad (15)$$

- Which states that the projected gradient is equal to zero at optimal. Similarly, the second order necessary condition for an optimal becomes

$$Z' \nabla_{xx}^2 f(x^*) Z \text{ is negative semi-definite} \quad (16)$$