

Lecture IX: Weierstrass's Theorem

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A Review of Expansions

- First we review of the general Taylor series expansion.
- Most have been introduced to the general concept that Taylor series expansion is an approximation of a general function $f(z)$ at some point (z_0) .
- In univariate space, this approximation is written as

$$f(z) = f(z_0) + \left. \frac{\partial f(z)}{\partial z} \right|_{z=z_0} (z - z_0) + \frac{1}{2} \left. \frac{\partial^2 f(z)}{\partial z^2} \right|_{z=z_0} (z - z_0)^2 + \dots \frac{1}{n!} \left. \frac{\partial^n f(z)}{\partial z^n} \right|_{z=z_0} (z - z_0)^n + \dots \quad (1)$$

- This expression is exact in the infinite sum.
- The infinite form is sometimes finite.
- For example, if $f(z)$ is a quadratic production function, the expression is exact.

$$f(x) = x^2 + ax + c \quad (2)$$

- Using the Taylor apparatus, we approximate the value of the function at x_1 given the value of the function at x_0

$$\begin{aligned}
 f(x) &= f(x_0) + (2x_0 + a)(x - x_0) + \frac{2}{2}(x_1 - x_0)^2 \\
 &= \{x_0^2 + ax_0 + c\} + \{2x_0x + ax - 2x_0^2 - ax_0\} + \{x^2 - 2xx_0 + x_0^2\} \\
 &= x_1^2 + ax_1 + c
 \end{aligned}
 \tag{3}$$

- So in the case of the quadratic function, the second order Taylor series expansion is exact.

- The more mathematically interesting fact is that if the sum in Equation 1 is truncated, the residual can be bounded

$$f(z) = f(z_0) + \left. \frac{\partial f(z)}{\partial z} \right|_{z=z_0} (z - z_0) + \frac{1}{2} \left. \frac{\partial^2 f(z)}{\partial z^2} \right|_{z=z_0} (z - z_0)^2 + \dots \frac{1}{n!} \left. \frac{\partial^n f(z)}{\partial z^n} \right|_{z=z_0} (z - z_0)^n + R_{n+1}$$

$$\text{where } R_{n+1} = \frac{1}{(n+1)!} \left. \frac{\partial^{n+1} f(z)}{\partial z^{n+1}} \right|_{z=z_0} (z^* - z_0)^{n+1} \text{ for some } z^* \in [z, z_0] \quad (4)$$

- Equation 1 states that the infinite order expansion is an equivalent function to the original function while Equation 4 depicts the approximation over the first n derivatives.
- This approximation has an error, but the error is bounded based on the $n + 1$ derivative.
- However, it is possible that the $n + 1$ derivative does not exist.

- In order to induce an error term, consider the second order expansion of a cubic function

$$g(x) = x^3 + ax^2 + bx + c$$

$$\Rightarrow g(x) = g(x_0) + (3x_0^2 + 2ax_0 + b)(x - x_0) + \quad (5)$$

$$\frac{6x_0 + 2a}{2}(x^* - x_0) \text{ for some } x^* \in [x, x_0]$$

- The second order Taylor series expansion is used to approximate unknown production functions.

- Other useful expansions include the Fourier expansion which depicts a function as the sum of sinusoids:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos(w_n t) + b_n \sin(w_n t)] \quad (6)$$

- Where w_i is a specific frequency. The notion is that a sufficient number of sinusoids can be added together to sufficiently proxy any general real function.

- The real workhorse in our discussion of optimality is the Taylor series expansion.
 - Assume that there exists a univariate function $f(x)$ defined over a given range $x \in [x_0, x_1]$.
 - x^* is defined as a strong local minimum if $f(x^*) < f(x)$ for all $x \in N(x^*, \delta)$, $x \neq x^*$. x^* is a weak local minimum if $f(x^*) \leq f(x)$ for all $x \in N(x^*, \delta)$, $x \neq x^*$.
 - Local optimals are not sufficient (i.e. additional assumptions are required to guarantee that local optimals are also global).

- Next, we derive the conditions for optimality from Taylor series.
- First, note that any point $x \in N(x^*, \delta)$ can be derived as a Taylor series expansion of $f(x^*)$

$$f(x) = f(x^*) + f'(x^*)(x - x^*) + \frac{1}{2}f''(x^*)(x - x^*)^2 + R_3 \quad (7)$$

- Assuming that $R_3 \rightarrow 0$ as $\delta \rightarrow 0$ and rearranging we have

$$f(x) - f(x^*) = f'(x^*)(x - x^*) + \frac{1}{2}f''(x^*)(x - x^*)^2 \quad (8)$$

- If $f(x^*) < f(x)$ for all $x \in N(x^*)$ then $f(x) - f(x^*) > 0$ for $x \in N(x^*)$.

- Thus, if x^* is a strong local minimum then

$$f'(x^*)(x - x^*) + \frac{1}{2}f''(x^*)(x - x^*)^2 > 0 \quad (9)$$

- Given that $x \in N(x^*)$ is a two sided concept $(x - x^*) \geq 0$.
- Thus, as $x \rightarrow x^*$ from the left $f'(x)$. Similarly, if $x \rightarrow x^*$ from the right $f'(x) > 0$.

- If the first derivative is smooth, these conditions imply that $f'(x^*) = 0$ in order for the derivative to switch from negative to positive at x^* .
- In addition, if the first derivative is smooth so that the derivative switches from negative to positive, $f''(x^*) > 0$ for $x \in N(x^*)$.

Weierstrass's Theorem

The Weierstrass theorem guarantees that a continuous function has a maximum and a minimum on a closed and bounded interval.

The notion of a continuous function is that there exists a function defined on an interval A such that

$$f : A \rightarrow \mathbb{R}^1 \text{ where } A = [x_0, x_1] \text{ and } \lim_{x \rightarrow x^* -} f(x) = \lim_{x \rightarrow x^* +} f(x) = f(x^*) \quad (10)$$

for all $x^* \in A$. Thus, there are no holes in the functional mapping.

- Given this general structure we know that the maximum or minimum must either be at a local maximum or minimum, or at a boundary point.
- The notion that the function is continuous presents the possibility of discontinuous derivatives.
- Thus, we are left with two general scenarios.

- A continuous function on a closed and bounded interval does not possess a local maximum or minimum.
- Either $f(x^*) - f(x) < 0$ for all $x^* > x$ in the interval in which case the function is constantly decreasing over the entire interval.
 - Thus, function reaches its maximum at the lower bound of the interval and its minimum at the upper bound of the interval.

- Alternatively, $f(x^*) - f(x) > 0$ for all $x^* > x$ in the interval in which case the function is constantly increasing over the entire interval.
 - Hence, the function reaches its maximum at the upper bound of the interval and its minimum at the lower bound of the interval.
- If the function possesses local maxima or minima in the interval the global maximum or minimum may occur at either one of those local optima or at the boundaries.
- Note that we only guarantee the existence of the maximum or minimum values, not the uniqueness of x associated with those values.

Regularity Conditions

The smoothness of the calculus then allows for a more unique characterization. If $f(x) \in C^1$ then the relative minimum or maximum only occurs where $f'(x) = 0$. However, again, this does not guarantee a unique solution for x . Further, $f(x) \in C^2$ also does not guarantee a unique solution. However, if $f(x) \in C^2$ and $f''(x) < 0$ over the entire interval for a maximum or $f''(x) > 0$ over the entire interval for a minimum then the optima will be unique.