

Lecture XII: Example and NullSpace

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1 Example - Unconstrained Maximization

As an example, consider the optimization problem

$$\begin{aligned} \max_{y_1, y_2, x_1, x_2, x_3, x_4, x_5} & 10.0 \exp(y_1) + 15.0 \exp(y_2) - 1.50x_1 - 2.00x_2 \\ & - 1.25x_3 - 0.95x_4 - 0.50x_5 \\ \text{s.t. } y_1 = f_1(x_1, x_2, x_3, x_4, x_5) &= 0.075 \ln(x_1) + 0.125 \ln(x_2) + 0.025 \ln(x_3) \\ & + 0.050 \ln(x_4) + 0.225 \ln(x_5) \\ y_2 = f_2(x_1, x_2, x_3, x_4, x_5) &= 0.150 \ln(x_1) + 0.075 \ln(x_2) \\ & + 0.125 \ln(x_3) + 0.125 \ln(x_4) + 0.025 \ln(x_5) \end{aligned} \tag{1}$$

which is similar to maximizing the product of related Cobb-Douglas production functions. Substituting each of the constraints into the objective function yields

$$\begin{aligned} \max_{x_1, x_2, x_3, x_4, x_5} & 10.0x_1^{0.075}x_2^{0.125}x_3^{0.025}x_4^{0.050}x_5^{0.225} + \\ & 15.0x_1^{0.150}x_2^{0.075}x_3^{0.125}x_4^{0.125}x_5^{0.025} \\ & - 1.50x_1 - 2.00x_2 - 1.25x_3 - 0.50x_4. \end{aligned} \tag{2}$$

Consider the optimality of the point $x^* = \left\{ \begin{array}{ccccc} 0.8359 & 0.5659 & 0.6628 & 1.0239 & 2.8651 \end{array} \right\}$.
The gradient for the problem in Equation 2 are

$$\begin{aligned} \nabla_x \pi(x) &= \begin{bmatrix} -1.50 + \frac{0.9x_2^{0.075}x_3^{0.125}x_4^{0.125}x_5^{0.025}}{x_1^{0.85}} + \frac{0.375x_2^{0.125}x_3^{0.025}x_4^{0.050}x_5^{0.225}}{x_1^{0.925}} \\ -2.0 + \frac{0.45x_1^{0.150}x_3^{0.125}x_4^{0.125}x_5^{0.025}}{x_2^{0.925}} + \frac{0.625x_1^{0.075}x_2^{0.025}x_4^{0.050}x_5^{0.225}}{x_2^{0.875}} \\ -1.25 + \frac{0.750x_1^{0.150}x_2^{0.075}x_4^{0.125}x_5^{0.025}}{x_3^{0.875}} + \frac{0.125x_1^{0.075}x_2^{0.125}x_4^{0.050}x_5^{0.225}}{x_3^{0.950}} \\ -0.95 + \frac{0.750x_1^{0.150}x_2^{0.075}x_3^{0.125}x_5^{0.125}}{x_4^{0.875}} + \frac{0.25x_1^{0.075}x_2^{0.0125}x_3^{0.025}x_5^{0.225}}{x_4^{0.950}} \\ -0.50 + \frac{0.15x_1^{0.150}x_2^{0.075}x_3^{0.125}x_4^{0.125}}{x_5^{0.975}} + \frac{1.125x_1^{0.075}x_2^{0.125}x_3^{0.050}x_4^{0.050}}{x_5^{0.775}} \end{bmatrix} \\ &= \begin{bmatrix} 0.00002607 \\ 0.0000304 \\ 0.0000135 \\ -0.0000060 \\ 0.0008330 \end{bmatrix}. \end{aligned} \tag{3}$$

2 Linearly Constrained Optimum

Remembering from last lecture, the differential with the null space imposed becomes

$$f(x) - f(x^*) = \nabla_x f(x^*) Zp + \frac{1}{2} p' Z' \nabla_{xx}^2 f(x^*) Zp \leq 0 \tag{4}$$

for any p . The first necessary condition for a maximum becomes

$$\nabla_x f(x^*) Z = 0 \tag{5}$$

which states that the projected gradient is equal to zero at optimal. Similarly, the second order necessary condition for an optimal becomes

$$Z' \nabla_{xx}^2 f(x^*) Z \text{ is negative semi-definite} \tag{6}$$

Let's start from a simple utility maximization problem

$$\begin{aligned} \max_{x_1, x_2, x_3, x_4, x_5} u(x) &= x_1^{0.15} x_2^{0.20} x_3^{0.03} x_4^{0.13} x_5^{0.23} \\ \text{s.t. } 5x_1 + 2x_2 + 3x_3 + 4x_4 + x_5 &\leq 100 \end{aligned} \quad (7)$$

The point of potential optimum is $x^* = \left\{ \begin{matrix} 4.0541 & 13.5135 & 1.3514 & 4.3919 & 31.0811 \end{matrix} \right\}$.
At this point, the gradient of the utility function is

$$\nabla_x u(x^*) = \begin{bmatrix} 0.04142 \\ 0.08286 \\ 0.12429 \\ 0.16572 \\ 0.041430 \end{bmatrix} \quad (8)$$

So we need to construct the null space.

3 Canonical Form and Null Spaces

To demonstrate the meaning of null space matrices, I first want to discuss the use of canonical forms. A canonical form is a solution to an underidentified system of equations. For example in a crop selection model we may have a land and a labor constraint

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 100 \\ 2x_1 + 3x_2 + x_3 + 4x_4 + x_5 &= 250 \end{aligned} \quad (9)$$

The matrix operations used to reduce this matrix to a row reduced form are

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & | & 100 \\ 2 & 3 & 1 & 4 & 1 & | & 250 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & | & 100 \\ 0 & 1 & -1 & 2 & -1 & | & 50 \end{pmatrix} \quad (10)$$

The operation in Equation 10 subtracts 2 times row one from row two. The leading element of the second row of Equation 10 is a 1 (or the next matrix operation would be to multiply by an identity matrix replacing the second element of the second row with the inverse of the leading term. However, given the result on the right hand side of Equation 10 the goal is to subtract the second row from the first

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & | & 100 \\ 0 & 1 & -1 & 2 & -1 & | & 50 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & -1 & 2 & | & 50 \\ 0 & 1 & -1 & 2 & -1 & | & 50 \end{pmatrix} \quad (11)$$

The result on the right hand side of Equation 11 is a canonical form representing a solution to the system of equations. Specifically, $x_1 = 50$, $x_2 = 50$, $x_3 = 0$, $x_4 = 0$, and $x_5 = 0$ solves the system of equations. In addition, the solution says something about maintaining feasibility in the nonbasic variables. Taking the columns from the nonbasic, or zero-level, variables we can form a homogeneous set of equations ($Ax = 0$)

$$\begin{aligned} 2x_3 - x_4 + 2x_5 &= 0 \\ -x_3 + 2x_4 - x_5 &= 0 \end{aligned} \quad (12)$$

This system has the reduce form

$$\begin{aligned} x_3 + x_5 &= 0 \\ x_4 &= 0 \end{aligned} \quad (13)$$

so any solution where $x_3 = -x_5$ maintains feasibility, or satisfies the equations.

A technical definition of the *null space* is the set of solutions of the linear equations $Ax = 0$. However, a fuller understanding of the null space involves a discussion of dimension. To define the system of linear equations more rigorously

$$Ax = b \Rightarrow \begin{cases} A \in M_{n \times m} \\ x \in M_{m \times 1} \\ b \in M_{n \times 1} \end{cases} \quad (14)$$

where M denotes the set of matrices (or linear spaces). In the common parlance we have a system of n equations with m unknown values. In the most interesting null space $n < m$ or the system is underidentified (or not all the levels of the variables are determined by a given level of right hand sides $[b]$). The system is overidentified if $n > m$ which implies that the system is potentially inconsistent (i.e., not all the equations or constraints can be solved with the same set of variables $[x]$).

In order to "solve" this system (assuming that the rows of A are linearly independent) we know that we must have at least m nonzero variables in the canonical form. *The null space of the matrix A is the same dimension as*

the number of free variables. Taking the first three variables from the above example

$$\begin{aligned}x_1 + x_2 + x_3 &= 100 \\ 2x_1 + 3x_2 + x_3 &= 250\end{aligned}\tag{15}$$

the reduced form for Equation 15 is

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 50 \\ 0 & 1 & -1 & 50 \end{array} \right)\tag{16}$$

Notice that any change

$$\begin{aligned}\Delta x_1 &+ 2\Delta x_3 &= 0 \\ \Delta x_2 &- \Delta x_3 &= 0\end{aligned}\tag{17}$$

is still feasible. For example, if $\Delta x_3 = 10 \Rightarrow \Delta x_1 = -20$ and $\Delta x_2 = 10$. Then $x_3 + \Delta x_3 = 10$, $x_1 + \Delta x_1 = 30$, and $x_2 + \Delta x_2 = 60$ yieldin

$$\begin{aligned}x_1 + x_2 + x_3 &= 30 + 60 + 10 = 100 \\ 2x_1 + 3x_2 + x_3 &= 60 + 180 + 10 = 250\end{aligned}\tag{18}$$

The null space is basically a line in 3 space which shows how the variables can be traded to maintain feasibility.

Expanding to the first four variables of the original model

$$\begin{aligned}x_1 + x_2 + x_3 + x_4 &= 100 \\ 2x_1 + 3x_2 + x_3 + 4x_4 &= 250\end{aligned}\tag{19}$$

which yields a reduced form

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 50 \\ 0 & 1 & -1 & 2 & 50 \end{array} \right)\tag{20}$$

Again we note that

$$\begin{aligned}\Delta x_1 &+ 2\Delta x_3 &- \Delta x_4 &= 0 \\ \Delta x_2 &- \Delta x_3 &+ 2\Delta x_4 &= 0\end{aligned}\tag{21}$$

In this case, the null space is a plane in 4 space. If $\Delta x_3 = 1$ and $\Delta x_4 = 5$ the

$$\begin{aligned}\Delta x_1 &= -2\Delta x_3 + \Delta x_4 = 3 \\ \Delta x_2 &= \Delta x_3 - 2\Delta x_4 = -9\end{aligned}\tag{22}$$

which maintains feasibility.

3.1 Singular Value Decomposition

Checking in your linear algebra book, any matrix can be decomposed into parts

$$A = U W V' \quad (23)$$

Using Householder transformations (i.e., the sequence of elementary row operations in matrix form required to reduce the matrix to an identity). For example, the A matrix in this example can be decomposed into

$$A = \begin{bmatrix} 0.34396 & -0.93899 \\ 0.93899 & 0.34396 \end{bmatrix} \begin{bmatrix} 5.91856 & 0 & 0 & 0 & 0 \\ 0 & 0.985197 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0.37542 & -0.25484 & -0.81650 & -0.18257 & -0.30679 \\ 0.53407 & 0.094284 & 0.408245 & -0.73030 & -0.076670 \\ 0.216766 & -0.60397 & 0 & 0 & 0.75597 \\ 0.69272 & 0.44341 & 0 & 0.54772 & 0.15339 \\ 0.21677 & -0.60397 & 0.40825 & 0.36515 & -0.53688 \end{bmatrix} \quad (24)$$

Peeling off the last three columns of V' (the ones associated with the zero matrix of W)

$$\text{NullSpace}(A) = \begin{bmatrix} -0.81650 & -0.18257 & -0.30679 \\ 0.408245 & -0.73030 & -0.076670 \\ 0 & 0 & 0.75597 \\ 0 & 0.54772 & 0.15339 \\ 0.40825 & 0.36515 & -0.53688 \end{bmatrix} \quad (25)$$

Next, a new x can be generated from the original solution

$$x^* = \begin{bmatrix} 50 \\ 50 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -0.81650 & -0.18257 & -0.30679 \\ 0.408245 & -0.73030 & -0.076670 \\ 0 & 0 & 0.75597 \\ 0 & 0.54772 & 0.15339 \\ 0.40825 & 0.36515 & -0.53688 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 44.2669 \\ 48.89 \\ 2.3009 \\ 2.6511 \\ 1.8912 \end{bmatrix} \quad (26)$$

Returning to the utility function $A = \begin{pmatrix} 5 & 2 & 3 & 4 & 1 \end{pmatrix}$ the last four columns of the V' matrix are

$$\text{NullSpace}[A] = V' = \begin{bmatrix} -0.196116 & -0.605228 & -0.324102 & -0.188814 \\ 0 & 0 & 0 & 0.96295 \\ 0 & 0 & 0.907485 & -0.113288 \\ 0 & 0.786796 & -0.259281 & -0.151051 \\ 0.980581 & -0.121046 & -0.0648204 & -0.0377627 \end{bmatrix} \quad (27)$$

Multiplying this NullSpace matrix with the gradient from the utility function yields $\nabla_x u(x) V' = \begin{bmatrix} 0.0325 & 0.1003 & 0.0537 & 0.0313 \end{bmatrix}$.