

Externalities and Environmental Goods: Lecture XXIII

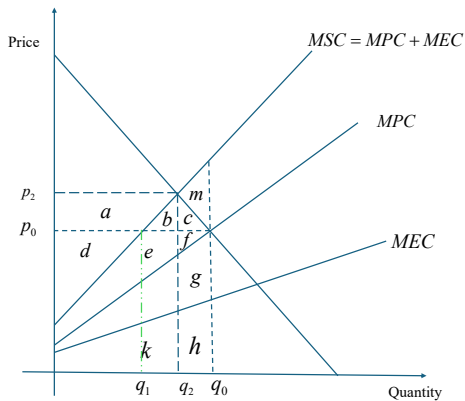
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- 1 Externality and the Social Optimal
- 2 WTP - Restricted Consumption of Goods

Externality and the Social Optimal

- I will take a fairly standard “Pigouvian” approach to this topic and then come back with the “Coasian” counter assertion.
- As depicted in Figure 4 this market is characterized by a departure of the marginal private cost (MPC) from the marginal social cost (MSC). The difference is the marginal externality cost (MEC).
- Following the standard discussion, without intervention of some sort the more output is produced than socially optimal (i.e., (q_0, p_0)).
- The socially optimal level of production is instead at (q_2, p_2) .



- Externalities exist in a variety of cases. The standard discussion usually centers around environmental externalities, but there can be externalities across profit maximizing firms.
- As a first step, consider the production function for two firms

$$\begin{aligned} q_1 &= x_1^{\frac{1}{2}} \\ q_2 &= x_2^{\frac{1}{2}} - x_0 \end{aligned} \tag{1}$$

where (x_1, q_1) are the amount of input used and output produced by the first firm, and (x_2, q_2) is the amount of input used and output produced by the second firm.

- Note that x_0 is a factor of production that the second firm takes as a given.
- The interesting part is where we set $x_0 = x_1^{\frac{1}{2}}$ so that the actions of the first firm effects the production of the second firm.

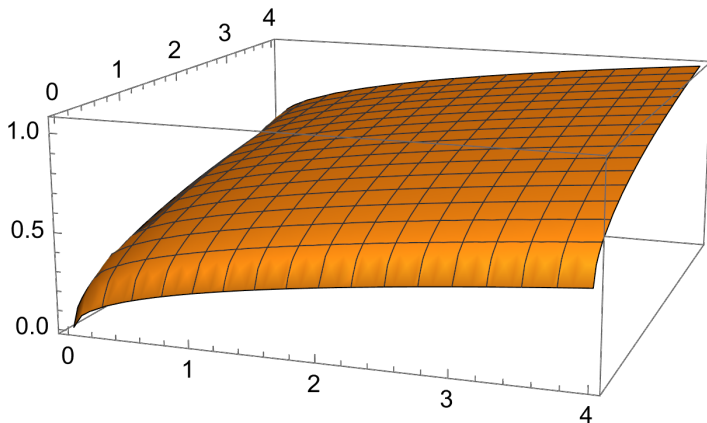
- As a general note, these production functions are known as the generalized square-root function or the Allen-McFadden-Samueulson (AMS) production Function (Diewert, 1971)

$$f(x_1, x_2) = \begin{bmatrix} 0.25 \\ 0.35 \end{bmatrix}' \begin{bmatrix} x_1^{\frac{1}{2}} \\ x_2^{\frac{1}{2}} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_1^{\frac{1}{2}} \\ x_2^{\frac{1}{2}} \end{bmatrix}' \begin{bmatrix} -0.0500 & 0.0025 \\ 0.0025 & -0.0150 \end{bmatrix} \begin{bmatrix} x_1^{\frac{1}{2}} \\ x_2^{\frac{1}{2}} \end{bmatrix} \quad (2)$$

This function is presented in Figure 7.

- If we let $p_y = 1.25$, $w_1 = 0.15$, and $w_2 = 0.25$ we will get an optimal profit of 0.3213 with $x_1^* = 0.9533$ and $x_2^* = 0.5952$.

Allen-McFadden-Samuelson Production Function



- Starting with a little simple calculus, the optimal level of x_1 is derived by

$$\max_{x_1} p_1 x_1^{\frac{1}{2}} - w_1 x_1 \Big\} \Rightarrow \begin{cases} x_1^{-\frac{1}{2}} = \frac{2w_1}{p_1} \\ x_1 = \left(\frac{p_1}{2w_1} \right)^2 \\ q_1 = \frac{p_1}{2w_1} \end{cases} \quad (3)$$

- Similarly,

$$q_2 = \frac{p_2 - 2w_2 x_1^{\frac{1}{2}}}{2w_2} \quad (4)$$

So the optimal production of firm 2 depends on the actions of firm 1.

- So what is the true cost (or Social Cost) of using an additional unit of x_1 , it is $w_1 + p_2 x_1^{\frac{1}{2}}$. Basically, every unit of x_1 used causes the profit of firm 2 to decline by $p_2 x_1^{\frac{1}{2}}$

$$\pi_2 = p_2 \left(x_2^{\frac{1}{2}} - x_1^{\frac{1}{2}} \right) - w_1 x_2 \quad (5)$$

- To work out the marginal cost we use the fact that $q_1 = x_1^{\frac{1}{2}}$ and substitute this into the standard cost specification

$$\begin{aligned} C_1(q_1) &= w_1 x_1 \ni x_1 = q_1^2 \Rightarrow C_1(q_1) = w_1 q_1^2 \\ &\Rightarrow \frac{\partial C_1(q_1)}{\partial q_1} = 2w_1 q_1 \\ \tilde{C}_1(q_1) &= w_1 x_1 + p_2 x_1^{\frac{1}{2}} \Rightarrow \tilde{C}_1(q_1) = w_1 q_1^2 + p_2 (q_1^2)^{\frac{1}{2}} \\ &\frac{\partial \tilde{C}_1(q_1)}{\partial q_1} = 2w_1 + p_2 \end{aligned} \quad (6)$$

- The Social Optimal would be achieved by “internalizing” the externality

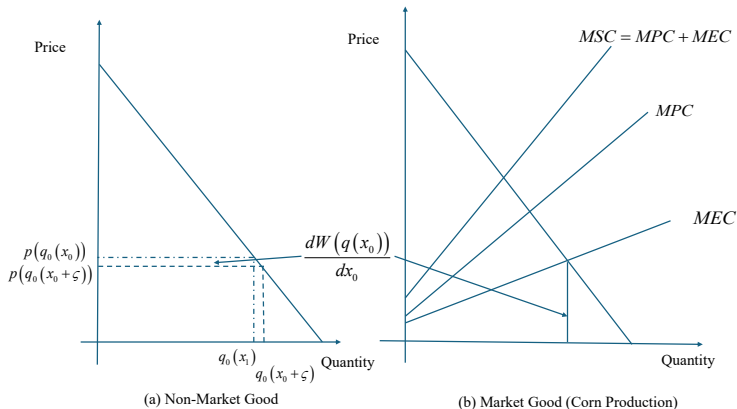
$$\begin{aligned}\tilde{\pi} &= p_1 q_1 - [w_1 q_1^2 + p_2 q_1] \\ \frac{\partial \tilde{\pi}}{\partial q_1} &= p_1 - 2w_1 q_1 - p_2 \\ q_1^* &= \frac{p_1 - p_2}{2w_1}\end{aligned}\tag{7}$$

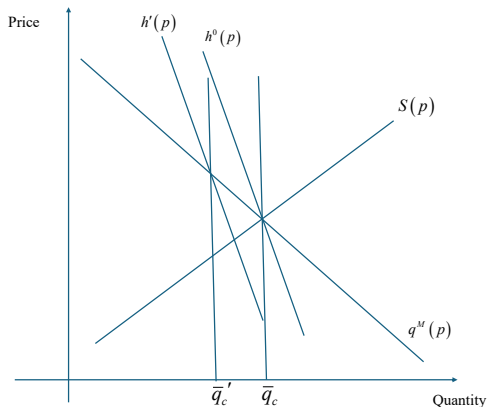
The same result can be obtained by solving the sum of the profit functions

$$\begin{aligned}\pi_T &= p_1 q_1 + p_2 q_2 - w_1 x_1 - w_1 x_2 \\ &= p_1 x_1^{\frac{1}{2}} + p_2 \left(x_2^{\frac{1}{2}} - x_1^{\frac{1}{2}} \right) - w_1 x_1 - w_2 x_2 \\ \frac{\partial \pi_T}{\partial x_1} &= \frac{1}{2} p_1 x_1^{-\frac{1}{2}} - \frac{1}{2} p_2 x_1^{-\frac{1}{2}} - w_1 = 0 \\ \Rightarrow \frac{1}{2} (p_1 - p_2) x_1^{-\frac{1}{2}} - w_1 &= 0 \\ \frac{\partial \pi_T}{\partial x_2} &= \frac{1}{2} x_2^{-\frac{1}{2}} - w_2 = 0\end{aligned}\tag{8}$$

- Returning to Figure 4 - at the original equilibrium (q_0, p_0) the cost of the externality is $f + h$. The socially optimum output at this price is q_1 - where the $MSC = p_1$.
- Equating the MSC with the demand curve yields an optimal price of p_2 and quantity of q_2 .
 - At this point production costs decrease by $g + h$,
 - Total revenue decreases by $f + g + h - a - b$,
 - Producers are better off by $a + b - f$,
 - Consumers lose $a + b + c$,
 - The gain from the reduction in the externality is h , and
 - The net social gain is $h - c - f = m$.

WTP - Restricted Consumption of Goods





- We start by considering the utility function with a resource endowment (r possibly zero for some goods) and a limited purchasable consumption level ($\bar{q}_C$)

$$\begin{aligned} \max_q U(q + r) \\ \tilde{m} = pq \end{aligned} \tag{9}$$

$$q_c = \bar{q}_c, q_u \geq 0, q(q_c, q_u)$$

- Based on this utility function, we can envision the expenditure function

$$\begin{aligned} \tilde{e}(p, \bar{U}, \bar{q}_c) &= \min_q p'q \\ U(q + r) &= \bar{U} \\ q_c &= \bar{q}_c, q_u \geq 0, q = (q_c, q_u) \end{aligned} \tag{10}$$

- Reworking Equation 10 slightly

$$\left. \begin{array}{l} \min_{q_u} p'_u q_u \\ U((\bar{q}_c, q_u) + r) = \bar{U} \\ q_u \geq 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \tilde{e}(p, \bar{U}, \bar{q}_c) = \\ p_c \bar{q}_c + p_u q_u(p, \bar{U}, \bar{p}_c) \end{array} \right. \quad (11)$$

- In order to make this a little easier – we collect the constrained goods q_{cj} to be the first n goods $q_{cj} \ni j = 1, \dots, n$. The second set of goods q_i are then the second set $q_i \ni i = n + 1, \dots, N$.
- Given this collection, we can differentiate the expenditure function from Equation 11 as

$$\frac{\partial \tilde{e}}{\partial q_{cj}} = p_j + \sum_{i=n+1}^N p_i \frac{\partial q_i}{\partial q_{cj}}, \quad j = 1, \dots, n. \quad (12)$$

- To make this a little easier to work with, we return to the Lagrange multiplier

$$\begin{aligned}
 L &= \sum_{i=1}^N p_i q_i + \phi [U(q) - \bar{U}] - \sum_{i=1}^n \psi_i (q_{ic} - \bar{q}_{ic}) \\
 \Rightarrow &\begin{cases} \frac{\partial L}{\partial q_i} = \phi \frac{\partial U}{\partial q_i} + \psi_i, \quad i = 1, \dots, n \\ \frac{\partial L}{\partial q_i} = \phi \frac{\partial U}{\partial q_i} \end{cases} \\
 \Rightarrow &\frac{\partial \tilde{e}}{\partial q_{jc}} = \psi_j + \phi \sum_{i=1}^N \frac{\partial U}{\partial q_i} \frac{\partial q_i}{\partial q_{jc}} = \psi_j + \frac{d\bar{U}}{d\bar{q}_{cj}}
 \end{aligned} \tag{13}$$

Where for convenience we have assumed

$\partial q_j / \partial q_{jc} = \partial q_{cj} / \partial q_{jc} = 1$ and $\partial q_k / \partial q_{jc} = 0$ for all $k, j = 1, \dots, n, k \neq j$.

- The real point of this derivation is that $\psi_j(p, \bar{U}, \bar{q}_c)$ (implicitly defined in Equation 13) is the marginal value (i.e., the implicit price) of the constrained good over its market price – it is precisely the Hicksian compensated value (inverse of the demand function) for q_{cj} .
- Given this formulation, we can define the compensating variation as

$$\begin{aligned}
 C &= \tilde{m}_1 - \tilde{e}(p_1, U_0, \bar{q}_{c1}) \\
 &= \tilde{m}_1 - \tilde{m}_0 + \tilde{e}(p_0, U_0, \bar{q}_{c0}) - \tilde{e}(p_1, U_0, \bar{q}_{c1}) \\
 &= \Delta \tilde{m} + \sum_{i=1}^n q_{i1} (p_{i1} - p_{i0}) - \sum_{i=1}^N \int_{q_{i0}}^{q_{i1}} \psi_i(p_0, U_0, \tilde{q}_i(q_i)) dp_i - \\
 &\quad \sum_{i=n+1}^N \int_{p_{i0}}^{p_{i1}} \tilde{q}_I(\tilde{p}_i(p_i) U_0, \bar{q}_{ic}) dp_i
 \end{aligned}$$

(14)

- I know that this looks awful, but breaking it down into pieces note that

$$\int_{q_{i0}}^{q_{i1}} \psi_i(p_0, U_0, \hat{q}_i(q_i)) dp_i \quad (15)$$

is the area under the marginal value from the inverse Hicksian demand function for the constrained goods.

- Similarly,

$$\int_{p_{i0}}^{p_{i1}} \tilde{q}_i(\hat{p}_i(p_i), U_0, \bar{q}_{c1}) dp_i \quad (16)$$

is the area under the variable Hicksian demand functions.

Positive and Negative Externalities

- This study analyzes the decision to permit additional agricultural water withdraws by Adena Springs Ranch (Ranch) in Marion County, Florida.
- The Ranch requested the authority to pump 13.267 million gallons of water per day (mgd) in August 2012 as part of a plan to shift from traditional cow-calf production to supplying grass-fat, hormone-free beef to specialty markets.
- Within the context of cost–benefit analysis, the proposed pumping will have both negative and positive externalities.
 - Pumping from the deep Floridian aquifer may reduce the output of Category 1 springs, which are those springs with an output of more than 64.6.
 - However, additional pumping by the Ranch may create increased surface flows supporting additional recreational flows for canoeing and kayaking in eastern Marion County.

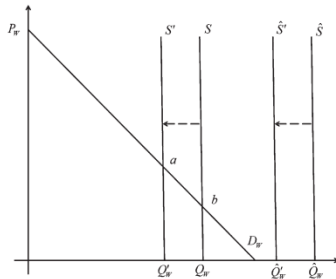


Figure 1. Effect of Additional Water Withdrawals on Environmental Flows

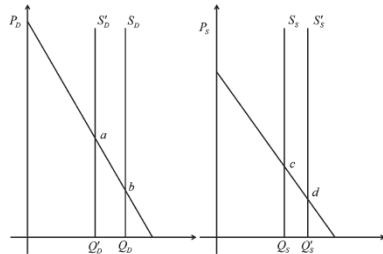


Figure 2. Attenuating Effect of Positive Externality

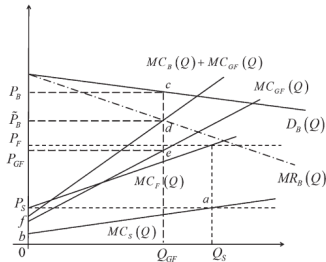


Figure 5. Market Equilibrium for Grass-Fat Cattle Operation

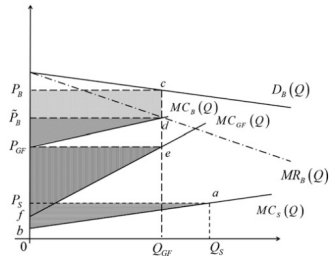


Figure 6. Producer and Consumer Surplus for Grass-Fat Beef

Gains to Cattle Operation (\$/Year)		
Cow-calf rents	297,225	
Processing rents	135,290	
Monopoly rents	912,946	
Grass fat beef—total rents		1,345,461
Gaines (Losses) from Environmental Flows)		
	Value/Day	
Value of ground water	(952.186)	(347,548)
Value of return flow	3.066	1119
Net Gaines (Losses)		
Net rents change		999,032
Original rents		(805,884)
Value of Change		193,148