

Effect of Exogenous Income on Factors of Production – Derivation of the Household Production Model: Lecture XVII

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An Example of the Income Effect from the Textbook

- As a starting point, consider the demand equation for the good produced by the producers

$$q = 11 - 5p + 2w + \frac{3}{50}\tilde{m} \quad (1)$$

and the quantity of labor supplied

$$h = -20 + 5w + 3p - \frac{2}{50}\tilde{m}. \quad (2)$$

- We will assume a starting point of $p_0 = 3$, $w_0 = 5$ and $\tilde{m}_0 = 100$.
- We want to consider a change where the government causes the price to decline to $w_1 = 4$, the price of consumption to decline to $p_1 = 2$ and the exogenous income to fall to $\tilde{m}_1 = 97$.

- The supply for labor that runs through the original point can be written as

$$h = -20 + 5w + 3 \times 3 - \frac{2}{50} \times 100 = -15 + 5w \quad (3)$$

Note that the original vector of prices $h = 10$, but when the price of labor falls to 4 $h = 5$. Therefore,

$$\Delta S_h = 5 \times (5 - 10) + (5 - 10) \times (4 - 5) = -\frac{15}{2}. \quad (4)$$

- In addition, we can define the income elasticity of labor supply as

$$\eta_h = \frac{\Delta h}{\Delta \tilde{m}} \frac{\tilde{m}}{h} = -\frac{2}{50} \frac{100}{h} = -\frac{4}{h} \quad (5)$$

- Considering the two endpoints of the observed labor supply curve $\nu_1^h = -4/10 = -0.4$ and $\nu_2^h = -4/5 = -0.8$.
- Next, we consider the product demand side of the economy

$$q = 11 - 5p + 2 \times 4 + \frac{3}{10}100 = 25 - 5p \quad (6)$$

Note that we used $w_1 = 4$ instead of $w_0 = 5$ – basically we are using a path. The overall change in surplus can then be derived as $\Delta S_q = 25/2$. In addition,

$$\eta_q = \frac{\Delta q}{\Delta \tilde{m}} \frac{\tilde{m}}{q} = \frac{3}{50} \frac{100}{q} = \frac{6}{q} \quad (7)$$

- Using a “sample” approach we start with an overall change in welfare of $\Delta S_h + \Delta S_q = -15/2 + 25/2 = 5$.
- However, we have to include the change in exogenous income $\tilde{m}_1 - \tilde{m}_0 = 97 - 100 = -3$. So the overall change in surplus is $\Delta S_h + \Delta S_q + \tilde{m}_1 - \tilde{m}_0 = 2$.

- Next, we turn to error of this approximation.
- Referring to Just et al.'s discussion of 'nibbling' on page 198, first note that the elasticity of demand for q_i with respect to exogenous income is labeled ρ_i for the i^{th} segment of the nibble, while the elasticity of the supply with respect to exogenous income is labeled $-\nu_i$.
- Given these definitions, we can "bound" the η s

$$\begin{aligned}\tilde{\eta}_1 &= \min(\rho_q, -\nu_h) = 0.4 \\ \tilde{\eta}_2 &= \max(\rho_q, -\nu_h) = 0.8\end{aligned}\tag{8}$$

Hence,

$$\tilde{s} = \frac{\Delta S_q + \Delta S_h}{\tilde{m}_0} = \frac{5}{100} = 0.05\tag{9}$$

- To wrap up the analysis

$$\begin{aligned}\eta_1 &= \frac{\tilde{\eta}_1 |\tilde{s}|}{2} = 0.01 \Rightarrow \tilde{C}_1 = 5 - 0.01 |5| = 4.95 \\ \eta_2 &= \frac{\tilde{\eta}_2 |\tilde{s}|}{2} = 0.02 \Rightarrow \tilde{C}_2 = 5 - 0.02 |5| = 4.90\end{aligned}\tag{10}$$

The Case with Consumption and Labor Supply

- I will refer to this general model as Becker's household production framework.
- What this model basically does is link the consumption model with the decision to work (or produce in the case of subsistence households).
- A key feature is the income constraint

$$wl + p'q = w\tau + \tilde{m} = m^*, \quad \tau \geq l \geq 0 \quad (11)$$

where w is the wage rate, l is the amount of leisure consumed, p is the price vector for good consumed, q is the quantity of good consumed.

- τ is the total time the individual has, so the amount of time worked is $\tau - l$.
- In this model, the cost of leisure if forgone wages wl .

- So under the second part of Equation 11, the total amount of money that the household spends (including leisure) is $w\tau + \tilde{m}$ where $\tilde{m}$ is the amount of exogenous income.
- So the household's model in terms of exogenous income can be written as

$$\begin{aligned}
 & \max_{l,q} U(l, q) \\
 & w(l - \tau) + p'q = \tilde{m} \\
 & \tau \geq l \geq 0, \quad q \geq 0 \\
 & \Rightarrow L = U(l, q) + \lambda (w(l - \tau) + p'q - \tilde{m})
 \end{aligned} \tag{12}$$

- Following through on this formulation

$$\left. \begin{aligned} U_q &= \lambda p \\ U_l &= \lambda w \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} q &= \tilde{q}(w, p, \tilde{m}) \\ l &= \tilde{l}(w, p, \tilde{m}) \end{aligned} \right. \tag{13}$$

- Where $\tilde{q}(w, p, \tilde{m})$ is the demand curves for the consumption goods, and $\tilde{l}(w, p, \tilde{m})$ is the demand for labor.
- The supply curve for labor can be written as

$$h = \tilde{h}(w, p, \tilde{m}) = \tau - \tilde{l}(w, p, \tilde{m}) \quad (14)$$

- In as slight extension of the traditional indirect utility function we can write

$$V = V(w, p, \tilde{m}) = U\left(\tilde{l}(w, p, \tilde{m}), \tilde{q}(w, p, \tilde{m})\right) \quad (15)$$

- Given this extension, we can derive the change in utility

$$\begin{aligned}
 \Delta U &= \int_L dU \\
 &= \int_L [U_l d\tilde{l} + U_q d\tilde{q}] \quad . \\
 &= \int_L \lambda [w d\tilde{l} + p d\tilde{q}]
 \end{aligned} \tag{16}$$

Where the last equation is derived by substituting the Lagrange term into the results (i.e., if $L = U + \lambda R$ where R is the restriction, then at optimum $dL \Rightarrow dU + \lambda dR = 0$; thus, $dU = \lambda dR$)

- Differentiating the income constraint

$$w d\tilde{l} + l dw + p d\tilde{q} + \tilde{q} dp = \tau dw + d\tilde{m} = dm^* \tag{17}$$

- Given that w and p are exogenous

$$\begin{aligned}
 \Delta U &= \int_L \lambda \left[d\tilde{m} - (\tilde{l} - \tau) dw - \tilde{q} dp \right] \\
 &= \int_L \lambda \left[d\tilde{m} + \tilde{h} dw - \tilde{q} dp \right] \\
 &= \int_L \left[dm^* - \tilde{l}^* dw - \tilde{q}^* dp \right]
 \end{aligned} \quad (18)$$

- As in our previous discussion with respect to the pure consumption case, we can divide the change in utility by λ to yield the change in surplus

$$\begin{aligned}
 \Delta S &= \int_L \left[d\tilde{m} + \tilde{h} dw - \tilde{q} dp \right] \\
 &= \int_L \left[dm^* - \tilde{l}^* dw - \tilde{q}^* dp \right]
 \end{aligned} \quad (19)$$

- Note the change in sign on the second term of the last equation in Equation 19. This is because the sign on leisure is the negative of the sign on labor.
- Just et al. (p. 219) gives the conditions required for the change in surplus to be independent of the path and conclude that these assumptions, like in the case of pure consumption, are unreasonable.
- One alternative is to make one of the goods a “numeraire” good which is fixed ($dp_N = 0$). “Hence, path independence is attained in the numeraire case when exogenous income changes if $\partial q/\partial m = 0$ for all goods other than the numeraire good.”

WTP Measures for Consumer-Laborers

- Given the development of the indirect utility function in the preceding section, it stands to reason that we can derive the Willingness-To-Pay measures from a similar expenditure function.

$$\tilde{e}(w, p, \tilde{U}) \Leftarrow \begin{cases} \min_{l, q} w(l - \tau) + p'q \\ U(l, q) = \tilde{U} \\ \tau \geq l \geq 0, q \geq 0 \end{cases} . \quad (20)$$

Note that Equation 20 is simply the dual of Equation 12 – it is the minimum exogenous income $\tilde{m}$ needed to generate utility U^* at the wage/price vector (w, p) .

- In addition, we can derive the compensated labor supply curve as

$$\begin{aligned}\tilde{e}_w &\equiv \tilde{e}_{p_0} = \tilde{l}(w, p, \tilde{U}) - \tau \\ &= \tilde{q}_0(w, p, \tilde{U}) - \tau \equiv -\tilde{h}(w, p, U^*)\end{aligned}\quad (21)$$

Where q_0 denotes the quantity of labor.

- The compensating variation can then be defined as

$$\begin{aligned}C &= \tilde{m}_1 - \tilde{e}(w_1, p_1, U_0) \\ &= \tilde{m}_1 - \tilde{m}_0 + \tilde{e}(w_0, p_0, U_0) - \tilde{e}(w_1, p_1, U_0) \\ &= \Delta\tilde{m} - \int_L \sum_{j=0}^N \tilde{e}_{p_j}(w, p, U_0) dp_j\end{aligned}\quad (22)$$

- The Equivalent variation can be written as

$$\begin{aligned}
 E &= \tilde{e}(w_0, p_0, U_1) - \tilde{m}_0 \\
 &= \Delta \tilde{m} - \int_L \sum_{j=0}^N \tilde{e}_{p_j}(w, p, U_1) dp_j
 \end{aligned} \tag{23}$$

- Note that in the case of both the compensating and equivalent variation p_0 is the implicit wage rate – cost of labor. The remaining p_j $j = 1, \dots, N$ are the consumption goods.
- The compensating (equivalent) variation for a general wage-price-income change is uniquely measure by adding to the changes in the areas left of compensated labor supply plus all the changes in the areas left of the compensated demands for goods, where each demand or supply is evaluated at the initial (final) utility level but successively conditioned on previously considered wage or price changes*