

Willingness-to-Pay Measures for Consumers – Duality: Lecture XV

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1 A Review of Duality for the Consumer's Demand Functions

A Review of Duality for the Consumer's Demand Functions

- But Professor Moss, that was last semester. Yes, but I need the functions
- Starting with the trusty CES function, we can define the Lagrangian for the cost minimization problem as

$$\begin{aligned} L &= p_1 q_1 + p_2 q_2 + \mu \left(U - [a q_1^r + (1-a) q_2^r]^{\frac{1}{r}} \right) \\ \frac{\partial L}{\partial q_1} &= p_1 - \mu \left[\frac{1}{r} [a q_1^r + (1-a) q_2^r]^{\frac{1}{r}-1} (r a q_1^{r-1}) \right] = 0 \\ \frac{\partial L}{\partial q_2} &= p_2 - \mu \left[\frac{1}{r} [a q_1^r + (1-a) q_2^r]^{\frac{1}{r}-1} (r (1-a) q_2^{r-1}) \right] = 0 \\ \frac{\partial L}{\partial \mu} &= U^* - [a q_1^r + (1-a) q_2^r]^{\frac{1}{r}} = 0 \end{aligned} \tag{1}$$

- Taking the ratio of the first two first-order conditions yields

$$\frac{p_1}{p_2} = \frac{a q_1^{r-1}}{(1-a) q_2^{r-1}} \quad (2)$$
$$\Rightarrow q_1 = \kappa p_1^{\frac{1}{r-1}} p_2^{\frac{1}{1-r}} q_2$$

- Substituting the result of Equation 2 into the partial with respect to the Lagrange multiplier

$$\left[a \left(\kappa p_1^{\frac{1}{r-1}} p_2^{\frac{1}{1-r}} q_2 \right)^r + (1-a) q_2^r \right]^{\frac{1}{r}} = U^*$$
$$\left[a \kappa^r p_1^{\frac{r}{r-1}} p_2^{\frac{r}{1-r}} + (1-a) \right]^{\frac{1}{r}} q_2 = u^* \quad (3)$$
$$p_2^{\frac{1}{1-r}} \left[a \kappa^r p_1^{\frac{r}{r-1}} + (1-a) p_2^{\frac{r}{r-1}} \right]^{\frac{1}{r}} q_2 = U^*$$

- Finishing the derivation from Equation 3

$$q_2^*(p_1, p_2, U^*) = \frac{p_2^{\frac{1}{r-1}} U^*}{\left[a\kappa^r p_1^{\frac{r}{r-1}} + (1-a) p_2^{\frac{r}{r-1}} \right]^{\frac{1}{r}}}. \quad (4)$$

- Substituting this result into Equation 2

$$q_1^*(p_1, p_2, U^*) = \frac{\kappa p_1^{\frac{1}{r-1}} U^*}{\left[a\kappa^r p_1^{\frac{r}{r-1}} + (1-a) p_2^{\frac{r}{r-1}} \right]^{\frac{1}{r}}}. \quad (5)$$

- Turning to the expenditure function

$$e(p_1, p_2, U^*) = p_1 q_1^*(p_1, p_2, U^*) + p_2 q_2^*(p_1, p_2, U^*) \quad (6)$$

$$\begin{aligned}
 e &= p_1 \left[\frac{\kappa p_1^{\frac{1}{r-1}} U^*}{\left[a \kappa^r p_1^{\frac{r}{r-1}} + (1-a) p_2^{\frac{r}{r-1}} \right]^{\frac{1}{r}}} \right] + \\
 &\quad p_2 \left[\frac{p_2^{\frac{1}{r-1}} U^*}{\left[a \kappa^r p_1^{\frac{r}{r-1}} + (1-a) p_2^{\frac{r}{r-1}} \right]^{\frac{1}{r}}} \right] \\
 &= \frac{\left(\kappa p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}} \right) U^*}{\left[a \kappa^r p_1^{\frac{r}{r-1}} + (1-a) p_2^{\frac{r}{r-1}} \right]^{\frac{1}{r}}}
 \end{aligned} \tag{7}$$

- Which is identical to our result from inverting the indirect utility function in Lecture 14.

- More detail on the following slides is presented in “Constant Elasticity of Demand Utility and Measurement of Consumer Surplus.”
- We can derive the own price elasticity of both Marshallian demand curves as

$$\begin{aligned}\eta_{11} &= \frac{1}{r-1} \left[1 - \frac{\kappa r p_1^{\frac{r}{r-1}}}{\kappa p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}} \right] \\ \eta_{22} &= \frac{1}{r-1} \left[1 - \frac{r p_2^{\frac{r}{r-1}}}{\kappa p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}} \right]\end{aligned}\tag{8}$$

- As a starting point, a little Alchemy – I would like to generate a decent solution. In theory, I would like to talk about food (agricultural output) that is fairly inelastic. However, the CES function is not very stable at the point. So I will start by approximating the CES solution around the point where $p_1 = 1.50$ and $p_2 = 1.25$. I would like to solve for the values of a and r such that the elasticity of demand for good 1 (food) is -1.05 and the elasticity of demand for the second good is -1.55. Mathematically, I solve

$$\min_{a,r} \left(\frac{1}{r-1} \left[1 - \frac{\left(\frac{1-a}{a}\right)^{\frac{1}{r-1}} r p_1^{\frac{r}{r-1}}}{\left(\frac{1-a}{a}\right)^{\frac{1}{r-1}} p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}} \right] + 1.05 \right)^2 + \left(\frac{1}{r-1} \left[1 - \frac{r p_2^{\frac{r}{r-1}}}{\left(\frac{1-a}{a}\right)^{\frac{1}{r-1}} p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}} \right] + 1.25 \right)^2 \quad (9)$$

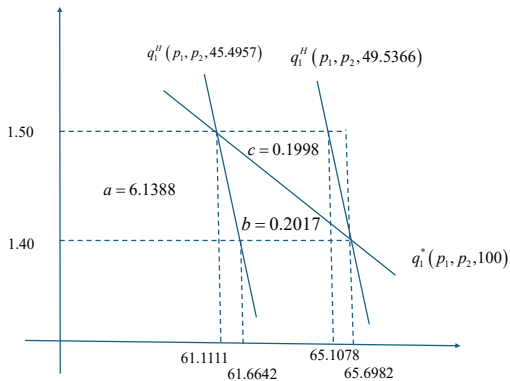
which yields $a \doteq 0.8274$ and $r \doteq 0.3750$

- Assuming an income of \$ 100 and plugging these values into Equation 2 yields $q_1^*(1.50, 1.25) = 61.1111$ and $q_2^*(1.50, 1.25) = 6.6667$. Plugging these values into the Indirect Utility function in Equation 3 yields $v(p_1, p_2, y) = 46.4957$. To complete the circle, we use the Hicksian demand from Lecture 15

$$q_1^H(p_1, p_2, U^*) = \frac{\left(\frac{1-a}{a}\right)^{\frac{1}{r-1}} p_1^{\frac{1}{r-1}} U^*}{\left[a \left(\frac{1-a}{a}\right)^{\frac{r}{r-1}} p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}} \right]^{\frac{1}{r}}} \quad (10)$$

where $q_1^H(1.5, 1.25, 46.4957) = 61.1111$ which is the same as the Marshallian demand.

- Next, consider what happens if we reduce the price of the first good to 1.40 – the Marshallian demand then becomes $q_1^*(1.40, 1.25, 100) = 65.6982$. I want to follow this through the discussion of compensating and equivalent variations.
 - The first step is to define indirect utility at the new price $v(1.40, 1.25, 100) = 49.5366$. As a “check” we compute the Hicksian demand for good 1 where $p_1 = 1.40$ and $U^* = 49.5366 \rightarrow q_1^H(.) = 65.6982$.
 - Next, we can solve for the Hicksian demand for the first good at the new utility and the old price $q_1^H(1.50, 1.25, 49.5366) = 65.1078$.
 - Going back to the original utility, the Hicksian demand at the original utility and the new price becomes $q_1^H(1.40, 1.25, 46.4957) = 61.6652$.



- Given the above derivation, we have demonstrated the first two conjectures of the appendix

$$\begin{aligned} e(p, U^*) &= p'q^*(p, U^*) \\ V(p, m) &= e^{-1}(p, m) \end{aligned} \quad (11)$$

- Consider a change in prices and income from (p_0, m_0) to (p_1, m_1) which results in an increase in utility from U_0 to U_1 .
 - The Hicksian compensating variation (C) – the amount of income that must be taken away from the consumer to restore the consumers original welfare level – can be defined as

$$\begin{aligned} V(p_1, m_1 - C) &= V(p_0, m_0) = U_0 \\ \Rightarrow C &= m_1 - e(p_1, U_0) \end{aligned} \quad (12)$$

- The Hicksian variation (E) is the amount of income the must be given to the consumer in lieu of price and income changes which leaves them as well off without the shift

$$\begin{aligned}U_1 &= V(p_1, m_1) = V(p_0, m_0 + E) \\ \Rightarrow E &= e(p_0, U_1) - m_0\end{aligned}\tag{13}$$

- Next, we are working toward the general form

$$\begin{aligned}e(p_0, U^*) - e(p_1, U^*) &= - \int_L e_p(p, U^*) dp \\ &= - \int_L q^h(p, U^*) dp\end{aligned}\tag{14}$$

- As a starting point, I would like to borrow a result from production - Shephard's lemma. We define the cost of production as the minimizing function $c(w, y) = \sum_{i=1}^n w_i x_i$. Given this definition, we define a function $g(w)$ as the difference

$$g(w, y) = c(w, y) - \sum_{i=1}^n w_i x_i^*(w, y) = 0$$

$$\frac{\partial g(w, y)}{\partial w_i} = \frac{\partial c(w, y)}{\partial w_i} - x_i^*(w, y) = 0$$
(15)

- Given the parallel between the expenditure function and the cost function

$$g(p, U) = e(p, U) - \sum_{i=1}^N p_i q_i^h(p, U) = 0$$

$$\frac{\partial g(p, U)}{\partial p_i} = \frac{\partial e(p, U)}{\partial p_i} - q_i^h(p, U) = 0$$
(16)

- Thus, we have demonstrated the last result in Equation 14.
- Because of the integrand in Equation 14 is an exact differential of e with respect to p no path-dependence problems are encountered.
- Working on a generalization (i.e., changing more than one price and changing the level of income)

$$\begin{aligned} C &= m_1 - e(p_1, U_0) \\ &= m_1 - m_0 + e(p_0, U_0) - e(p_1, U_0) \\ &= \Delta m - \int_L e_p(p, U_0) dp \\ &= \Delta m - \sum_{j=1}^N \int_{p_{0j}}^{p_{1j}} q_j(\tilde{p}_j(p_j), U_0) dp_j \end{aligned} \tag{17}$$