

Willingness-to-Pay - Inner and Outer Measures: Lecture XIV

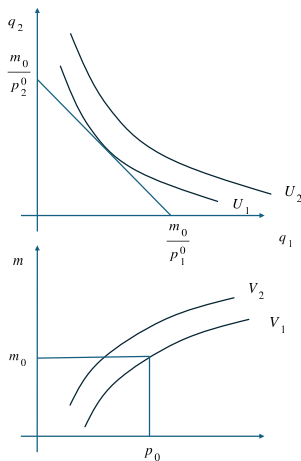
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Graphical Analysis of Willingness to Pay



Let's Do the Math

- Starting from a form of the *Constant Elasticity of Substitution* utility function

$$\begin{aligned}L &= [aq_1^r + (1-a)q_2^r]^{\frac{1}{r}} + \lambda(y - p_1q_1 - p_2q_2) \\ \frac{\partial L}{\partial q_1} &= \frac{1}{r} [aq_1^r + (1-a)q_2^r]^{\frac{1}{r}-1} (aq_1^{r-1}) - \lambda p_1 = 0 \\ \frac{\partial L}{\partial q_2} &= \frac{1}{r} [aq_1^r + (1-a)q_2^r]^{\frac{1}{r}-1} ((1-a)q_2^{r-1}) - \lambda p_2 = 0 \\ \frac{\partial L}{\partial \lambda} &= y - p_1q_1 - p_2q_2 = 0\end{aligned}\tag{1}$$

Note that solving the first two first-order conditions implies

$$\frac{aq_1^{r-1}}{(1-a)q_2^{r-1}} = \frac{p_1}{p_2}\tag{2}$$

- Solving Equation 2 for q_1

$$\begin{aligned}
 q_1 &= \left[\frac{1-a}{a} \right]^{\frac{1}{r-1}} \left[\frac{p_1}{p_2} \right]^{\frac{1}{r-1}} q_2 \\
 q_1 &= \kappa p_1^{\frac{1}{r-1}} p_2^{\frac{1}{1-r}} q_2 \\
 \ni: \kappa &= \left[\frac{1-a}{a} \right]^{\frac{1}{r-1}}
 \end{aligned} \tag{3}$$

- Substituting this result into the budget constraint

$$\begin{aligned}
 p_1 \left[\kappa p_1^{\frac{1}{r-1}} p_2^{\frac{1}{1-r}} \right] + p_2 q_2 &= y \\
 \left[\kappa p_1^{\frac{r-1+1}{r-1}} p_2^{\frac{1}{1-r}} + p_2 \right] q_2 &= y \\
 \left[\kappa p_1^{\frac{r}{r-1}} p_2^{\frac{1}{1-r}} + p_2 \right] q_2 &= y
 \end{aligned} \tag{4}$$

- Just a little flim/flam - There is a solution to the last equation in Equation 4 that solves

$$\begin{aligned}
 p_2^{\frac{1}{1-r}} \left[\kappa p_1^{\frac{r}{r-1}} + p_2^z \right] q_2 &= y \\
 \Rightarrow z &= 1 - \frac{1}{1-r} = \frac{1-r}{1-r} - \frac{1}{1-r} \Rightarrow z = \frac{r}{r-1} \\
 \therefore p_2^{\frac{1}{1-r}} \left[\kappa p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}} \right] q_2 &= y
 \end{aligned} \tag{5}$$

Therefore

$$q_2^*(p_1, p_2, y) = \frac{p_2^{\frac{1}{r-1}} y}{\left[\left(\frac{1-a}{a} \right)^{\frac{1}{r-1}} p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}} \right]}. \tag{6}$$

- Substituting Equation 6 into Equation 3

$$q_1^*(p_1, p_2, y) = \frac{\left(\frac{1-a}{a}\right)^{\frac{1}{r-1}} p_1^{\frac{1}{r-1}} y}{\left[\left(\frac{1-a}{a}\right)^{\frac{1}{r-1}} p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}\right]}. \quad (7)$$

- Substituting Equations 6 and 7 back into the utility function and collecting terms yields the indirect utility function

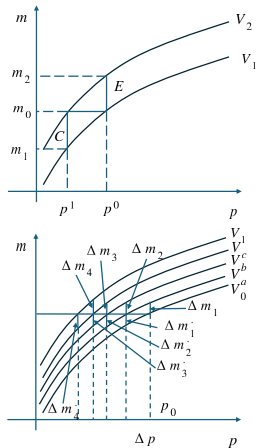
$$v(p_1, p_2, y) = \frac{\left[a \left(\frac{1-a}{a}\right)^{\frac{r}{r-1}} p_1^{\frac{r}{r-1}} + (1-a) p_2^{\frac{r}{r-1}}\right]^{\frac{1}{r}} y}{\left[\left(\frac{1-a}{a}\right)^{\frac{1}{r-1}} p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}\right]}. \quad (8)$$

- Inverting Equation 8 we derive the expenditure function

$$e(p_1, p_2, U^*) = \frac{\left[\left(\frac{1-a}{a} \right)^{\frac{1}{r-1}} p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{1-r}} \right] u^*}{\left[a \left(\frac{1-a}{a} \right)^{\frac{r}{r-1}} p_1^{\frac{r}{r-1}} + (1-a) p_2^{\frac{r}{r-1}} \right]^{\frac{1}{r}}} \quad (9)$$

- To verify the solution, I set $p_1 = 3$, $p_2 = 2.5$, $a = 0.3$, $r = 0.5$, and $y = 100$ and solved the constrained maximization problem using Mathematica which yielded $q_1 = 4.4248$, $q_2 = 34.690$ and $U^* = 22.6$. Plugging these values into Equations 7, 6, and 8 yield the same results. In addition, Equation 9 yields 100.

Compensating and Equivalent Variation through the Indirect Utility Function



- Consider the results from the *CES* utility function.
 - $v_1 = v(p_1 = 3, p_2 = 2.50, y = 100) = 22.6$.
 - $v_1 = v(p_1 = 2.75, p_2 = 2.50, y = 98.8076)$ which is the point (m_1, p_1) therefore $C = 1.1924$.
 - Going the other way, if we assume $v_2 = v[p_1 = 2.75, p_2 = 2.5, y = 98.8076] = 22.8727$, we can solve the expenditure function $e[p_1 = 3, p_2 = 2.5, u^* = 22.8727] = 101.207$ therefore $E = 1.2070$.
- Panel b, then demonstrates chopping the change up into segments
 - $C = \Delta m_1^* + \Delta m_2^* + \Delta m_3^*$.
 - $E = \Delta m_1 + \Delta m_2 + \Delta m_3 + \Delta m_4$.

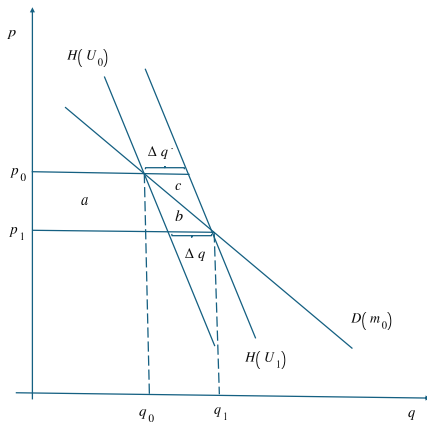
Slutsky Equation

- Varian [pp.130-131]

$$\begin{aligned}
 h_j(p, u^*) &= x_j(p, e(p, u^*)) \\
 \frac{\partial h_j(p^*, u^*)}{\partial p_i} &= \frac{\partial x_j(p^*, y^*)}{\partial p_i} + \frac{\partial x_j(p^*, y^*)}{\partial y} \frac{\partial e(p^*, u^*)}{\partial p_i} \\
 \Rightarrow \frac{\partial x_j(p^*, y^*)}{\partial p_i} &= \frac{\partial h_j(p^*, u^*)}{\partial p_i} - \frac{\partial x_j(p^*, y^*)}{\partial y} x_i^*
 \end{aligned} \tag{10}$$

Consumer Surplus as a WTP Measure: The Single-Price-Change Case

- As we previously discussed, the ordinary consumer surplus measure:
 - 1 is an intuitively appealing measure of consumer welfare effect of price and/or income changes,
 - 2 is generally an ambiguous (that is nonunique) measure,
 - 3 requires very strict restrictive path-independent conditions on the utility function for uniqueness, and
 - 4 requires even more restrictive conditions of constancy of the marginal utility of income to guarantee even an ordinal (quantitative) relationship with the actual marginal utility.
- Even though the ordinary consumer surplus cannot be justified directly, an approximate relationship exists with the WTP concepts under a broad range of conditions.



- The ordinary consumer surplus change ΔS , is the area $a + b$, and the compensating and equivalent variations, C and E_j are the areas a and $a + b + c$, respectively.
- Two important observations on these results:
 - First, if areas b and c are negligible, the consumer surplus change may be directly used as an approximation of both the compensating and equivalent variations.
 - Second, if areas b and c can be estimated from observable phenomena by some means, consumer surplus can be approximately modified to produce approximations of the compensating and equivalent variations.
- An estimate of areas b and c can be developed by noting that each is approximately a triangle for small price changes and that the height of each, Δp , is determined by the extent of the price change.
 - Willig, R.D. (1976) Consumer's Surplus Without Apology *American Economic Review* 66(4), 589-597.

- Also, the base of each triangle, Δq and Δq^* , is essentially an income effect, that is, the effect of changing income by a certain amount while holding the price fixed.
 - Note that this income effect is evaluated at two points – m_0/p_1 and m_0/p_0 .
- Using the elasticity formulas as we have done several times in preceding lectures

$$\eta = \frac{\Delta q}{\Delta m} \frac{m}{q} \quad (11)$$

$$\Delta q = \eta \times q \times \frac{\Delta m}{m}$$

- Under Willingness-to-Pay assumptions

$$\Delta m \doteq \Delta S. \quad (12)$$

- Substituting the result from Equation 12 into the last result from Equation 11

$$\Delta q \doteq \eta \times q \times \frac{\Delta S}{m} \quad (13)$$

- Therefore, since we know $\Delta p = p_0 - p_1$

$$b \doteq \frac{1}{2} \Delta q \times |\Delta p| \doteq \frac{1}{2} \eta \times \Delta S \times q \times \frac{|\Delta p|}{m}. \quad (14)$$

- Noting that $\Delta S \doteq q |\Delta p|$ for small Δp , the compensating variation can be approximated by subtracting the result from Equation 14 (for b) from the area $a + b$

$$\begin{aligned} \hat{C} &= \Delta S - \frac{\eta}{2m} (\Delta S)^2 = \Delta S - \hat{\epsilon} |\Delta S| \\ \hat{\epsilon} &= \frac{\eta |s|}{2}, \quad s = \frac{\Delta S}{m} \end{aligned} \quad (15)$$

- For the equivalent variation

$$\begin{aligned}\Delta q^* &\doteq -\eta \times q \times \frac{\Delta S}{m} \\ c &\doteq -\frac{1}{2}\Delta q^* \times |\Delta p| \doteq \frac{1}{2}\eta \times \Delta S \times q \times \frac{|\Delta p|}{m} \\ \hat{E} &= \Delta S + \frac{\eta}{2m} (\Delta S)^2 = \Delta S + \hat{\epsilon} |\Delta S|\end{aligned}\quad (16)$$

- A final reference to deriving the Hicksian demand from the Marshallian demand – Hausman, J.A. (1981) Exact Consumer's Surplus and Deadweight loss *American Economic Review* 71(4), 662 - 676.