

Finishing the Properties of Consumer Surplus: Lecture IX

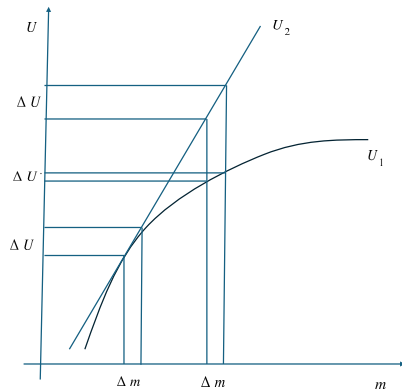
Andrew Schmitz and Charles B. Moss

January 17, 2026

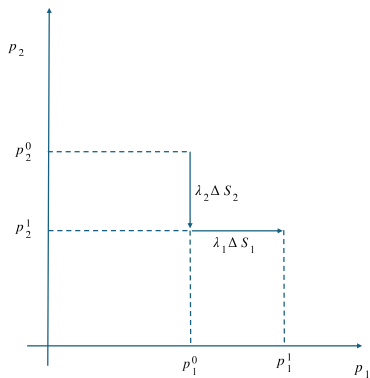
- 1 Constancy of the Marginal Utility of Income
- 2 The Pure Consumer Case

Constancy of the Marginal Utility of Income

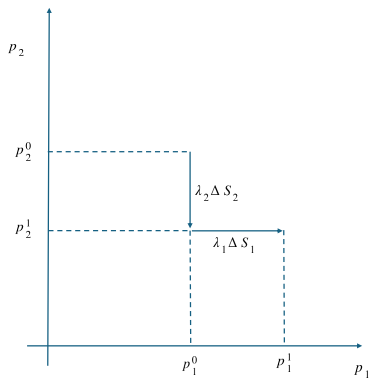
- The basic concept here is whether a change in consumer surplus yields a *money measure of utility change*
 - Under what conditions does a unique correspondence between the change in consumer surplus and the change in the consumer's actual utility exist?
- The “nuts and bolts” involves the concavity of utility functions.
- As depicted in Figure 4 we consider two utility functions U_1 and U_2 . Note that U_1 is concave while U_2 is linear.



- The basic concept is that the change in utility is different when the change in income takes place from two different income levels.
 - At the lower level of income, the change in utility is pretty close to one another.
 - However, at the higher level of income $\Delta U^* \ll \Delta U$. The change in utility is lower on the concave utility function
- The marginal utility is constant for U_2 .
- If the marginal utility is constant (U_2) there is a direct equivalence between change in surplus ΔS and the change in utility (ΔU).
 - Note that the implicit assumption in this figure is that only income changes - there is no change in prices.



- As a starting point, we assume that $\Delta S_2 = -\Delta S_1$.
- Next, we assume that the λ_1 is the marginal utility of income along the first section while λ_2 is the marginal utility of income along the second section.
- Hence, $\Delta U_1 = \lambda_1 \Delta S_1$ and $\Delta U_2 = \lambda_2 \Delta S_2$ - in each case the change in utility is equal to the change in surplus along a path.
- The difficulty is that if $\Delta S = \Delta S_1 + \Delta S_2 = 0$ - no change in surplus, $\Delta U \neq 0$ if $\lambda_1 \neq \lambda_2$.
- *Even with path independence, ΔS may not behave in a way qualitatively similar to ΔU .*
- *Uniqueness of ΔS is not sufficient to guarantee any meaningful interpretation of ΔS as a money measure of utility change.*



The Pure Consumer Case

- The basic consumption problem

$$\left. \begin{array}{l} \max_q U(q) \\ \text{s.t. } p'q \leq m \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} q(p, m) \\ U(q(p, m)) = V(p, m) = V \end{array} \right. \quad (1)$$

Hence, we can define the maximum utility based on a price vector and income level as the indirect utility function.

- Hence, the change in utility can be written based on the indirect utility function as

$$\begin{aligned}
 \Delta U &= V(p^1, m) - V(p^0, m) \\
 &= \int_L dV = \int_L [V_m dm + V_p dp] \\
 V_p dp &= \sum_{i=1}^N V_i dp_i \text{ and } V_M = \frac{\partial V}{\partial m} = \lambda
 \end{aligned} \tag{2}$$

- Where L is a path of integration.

- Noting that $dV = dU = U_q dq$

$$\Delta U = \int_L U_1 dq = \int_L \lambda p dq \text{ by optimality conditions } U_q = \lambda p$$

$$dm = qdp = p dq$$

$$\Delta U = \int_L \lambda [dm - qdp]. \quad (3)$$

- This line integral gives an exact measure of utility change.
- The point of interest is how to make Equation 3 a meaningful measure of welfare change.
- The usual approach has been to assume that λ is 'approximately' constant over the path L and thus divide Equation 3 by λ .

- Basically, we are interested in the expression

$$\Delta S = \int_L [dm - qdp] \quad (4)$$

If λ is constant along L , then $\Delta S = \Delta U / \lambda$. If λ is not constant, ΔS is not proportion to ΔU , but simply measures a particular imputed dollar gain assigned by the consumer to utility change.

- Starting with Roy's identity

$$\begin{aligned} q &= -\frac{V_p}{V_m} \\ \Rightarrow -\lambda q &= V_p \end{aligned} \quad (5)$$

given $V_m = \lambda$.

- Differentiating Equation 5 with respect to m

$$\begin{aligned}
 -\lambda q_m - q\lambda_m &= \lambda_p \\
 \lambda_p &\equiv \frac{\partial V_m}{\partial p} \equiv \frac{\partial^2 V}{\partial p \partial m} \equiv \frac{\partial V_p}{\partial m}
 \end{aligned} \tag{6}$$

- It is thus evident that λ cannot be constant with respect to all prices and income because $\lambda_p = 0$ and $\lambda_m = 0$ imply that $q_m = 0$, but the budget constraint implies that

$$1 = \frac{\partial m}{\partial m} = \frac{\partial pq}{\partial m} = pq_m \tag{7}$$

which cannot hold if $q_m = 0$.