

Closed Form of the Cobb Douglas Utility Function

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1 I would like to add some “basics” behind the result

$$V(p_1, m_1 - C) = V(p_0, m_0) \quad (1)$$

where C is the compensating variation. To do this the first step is to generate “closed form” expenditure and indirect utility functions. So, as a first step, we derive the expenditure function starting with the ratio of the first order conditions

$$\begin{aligned} \frac{p_1}{p_2} &= \frac{\alpha q_1}{\beta q_2} \\ q_1 &= \frac{\alpha p_2}{\beta p_1} q_2 \end{aligned} \quad (2)$$

Substituting this result into the utility constraint

$$\begin{aligned} U^* - \left[\frac{\alpha p_2}{\beta p_1} q_2 \right]^\alpha q_2^\beta &= 0 \\ q_2 &= \left[\frac{\beta p_1}{\alpha p_2} \right]^{\frac{\alpha}{\alpha+\beta}} U^{*\frac{1}{\alpha+\beta}} \end{aligned} \quad (3)$$

Substituting this results back into Equation 2

$$q_1 = \left[\frac{\alpha p_2}{\beta p_1} \right]^{\frac{\beta}{\alpha+\beta}} U^{*\frac{1}{\alpha+\beta}} \quad (4)$$

Substituting these results into the expenditure constraint

$$\begin{aligned}
e(p_1, p_2, U^*) &= \left[\left(\frac{\alpha p_2}{\beta p_1} \right)^{\frac{\beta}{\alpha+\beta}} p_1 + \left(\frac{\beta p_1}{\alpha p_2} \right)^{\frac{\alpha}{\alpha+\beta}} p_2 \right] U^{*\frac{1}{\alpha+\beta}} \\
&= \left[\left(\frac{\alpha}{\beta} \right)^{\frac{\beta}{\alpha+\beta}} p_2^{\frac{\beta}{\alpha+\beta}} p_1^{\frac{\alpha}{\alpha+\beta}} + \left(\frac{\beta}{\alpha} \right)^{\frac{\alpha}{\alpha+\beta}} p_1^{\frac{\alpha}{\alpha+\beta}} p_2^{\frac{\beta}{\alpha+\beta}} \right] U^{*\frac{1}{\alpha+\beta}} \quad . \quad (5) \\
&= \left[\left(\frac{\alpha}{\beta} \right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha} \right)^{\frac{\alpha}{\alpha+\beta}} \right] p_1^{\frac{\alpha}{\alpha+\beta}} p_2^{\frac{\beta}{\alpha+\beta}} U^{*\frac{1}{\alpha+\beta}}
\end{aligned}$$

Turning to the Marshallian demand functions, we substitute the result of Equation 2 into budget constraint

$$\begin{aligned}
y - \left[\frac{\alpha p_2}{\beta p_1} q_2 \right] p_1 - p_2 q_2 &= 0 \\
\frac{\beta y - \alpha p_2 q_2 - \beta p_2 q_2}{\beta} &= 0 \quad . \quad (6) \\
q_2 &= \frac{y\beta}{(\alpha + \beta) p_2}
\end{aligned}$$

Substituting this result back into Equation 2 yields

$$q_1 = \frac{y\alpha}{(\alpha + \beta) p_1}. \quad (7)$$

Substituting these results into the utility function yields the indirect utility function

$$v(p, y) = \frac{\alpha^\alpha \beta^\beta}{(\alpha + \beta)^{\alpha+\beta}} \frac{y^{\alpha+\beta}}{p_1^\alpha p_2^\beta}. \quad (8)$$

Note that we can “invert” Equation 8 as

$$y = \frac{\alpha + \beta}{\alpha^{\frac{\alpha}{\alpha+\beta}} \beta^{\frac{\beta}{\alpha+\beta}}} p_1^{\frac{\alpha}{\alpha+\beta}} p_2^{\frac{\beta}{\alpha+\beta}} U^{*\frac{1}{\alpha+\beta}} \quad (9)$$

This result is actually the same as the result in Equation 5, the difficulty is the first fraction. To see the equivalence can be seen in

$$\begin{aligned}
\frac{\alpha + \beta}{\alpha^{\frac{\alpha}{\alpha+\beta}} \beta^{\frac{\beta}{\alpha+\beta}}} &= \frac{\alpha}{\alpha^{\frac{\alpha}{\alpha+\beta}} \beta^{\frac{\beta}{\alpha+\beta}}} + \frac{\beta}{\alpha^{\frac{\alpha}{\alpha+\beta}} \beta^{\frac{\beta}{\alpha+\beta}}} \\
&= \frac{\alpha^{\frac{\alpha+\beta}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta}}}{\beta^{\frac{\beta}{\alpha+\beta}}} + \frac{\beta^{\frac{\alpha+\beta}{\alpha+\beta} - \frac{\beta}{\alpha+\beta}}}{\alpha^{\frac{\alpha}{\alpha+\beta}}} \\
&= \left(\frac{\alpha}{\beta}\right)^{\frac{\alpha}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}.
\end{aligned} \tag{10}$$

If we assume $\alpha = 0.35$ and $\beta = 0.45$, these functions become

$$\begin{aligned}
v(p_1, p_2, y) &= \frac{0.5780y^{0.80}}{p_1^{0.35} p_2^{0.45}}, \quad e(p_1, p_2, U^*) = 1.9844 p_1^{0.4375} p_2^{0.5625} U^{*1.25} \\
q_1(p_1, p_2, y) &= \frac{0.4375y}{p_1}, \quad q_1^H(p_1, p_2, U^*) = 0.8682 \left(\frac{p_2}{p_1}\right)^{0.5625} U^{*1.25} \\
q_2(p_1, p_2, y) &= \frac{0.5625y}{p_2}, \quad q_2^H(p_1, p_2, U^*) = 1.1162 \left(\frac{p_1}{p_2}\right)^{0.4375} U^{*1.25}
\end{aligned} \tag{11}$$

2 Next, we consider the welfare impacts of a decline in the price of p_1 from 1.25 to 1.15.

a We discussed two “dual” specifications for the variations in class

$$\begin{aligned}
V(\tilde{p}, \tilde{y} - C) - V(p, y) &= 0 \\
V(\tilde{p}, \tilde{y}) - V(p, y + E) &= 0
\end{aligned} \tag{12}$$

Taking $p = (p_1 = 1.25, p_2 = 0.75)$ and $\tilde{p} = (p_1 = 1.15, p_2 = 0.75)$ compute C (the compensating variation) and E (the equivalent variation).

b Defining $e(p, U)$ as the expenditure function at the original level of utility relate

$$e(p, U) - e(\tilde{p}, U) = - \int_L e_p(p, U) dp \tag{13}$$

to one of the variations (either compensating or equivalent). How do the two measures compare?