

Constant Elasticity of Demand Utility and Measurement of Consumer Surplus

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February 10, 2026

- As a starting point we define the Constant Elasticity of Substitution utility function as

$$U(q_1, q_2) = [aq_1^r + (1 - a)q_2^r]^{\frac{1}{r}} \quad (1)$$

- Starting with the Marshallian (i.e., income based demand system), in Lecture 14 we derived the Marshallian demand functions as

$$\begin{aligned} q_1^*(p_1, p_2) &= \frac{\left(\frac{1-a}{a}\right)^{\frac{1}{r-1}} p_1^{\frac{1}{1-r}} y}{\left(\frac{1-a}{a}\right)^{\frac{1}{r-1}} p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}} \\ q_2^*(p_1, p_2) &= \frac{p_2^{\frac{1}{r-1}} y}{\left(\frac{1-a}{a}\right)^{\frac{1}{r-1}} p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}} \end{aligned} \quad (2)$$

And the indirect utility function can then be derived as

$$v(p_1, p_2, y) = \frac{\left[a \left(\frac{1-a}{a}\right)^{\frac{r}{r-1}} p_1^{\frac{r}{r-1}} + (1-a) p_2^{\frac{r}{r-1}} \right]^{\frac{1}{r}} y}{\left(\frac{1-a}{a}\right)^{\frac{1}{r-1}} p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}} \quad (3)$$

We did not discuss the derivation of the slopes and elasticities of each of the Marshallian demand curves.

– Starting with good 1

$$\begin{aligned} \frac{\partial q_1}{\partial p_1} &= \frac{1}{r-1} \frac{\kappa p_1^{-1} p_1^{\frac{1}{r-1}} y}{\kappa p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}} - \kappa p_1^{\frac{r}{r-1}-1} \frac{r}{r-1} \frac{\kappa p_1^{\frac{1}{r-1}} y}{\kappa p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}} \\ \ni: \kappa &= \left(\frac{1-a}{a} \right)^{\frac{1}{r-1}} \end{aligned} \quad (4)$$

Note that we have left this derivation in a form where we can easily substitute q_1 into several parts of Equation 4

$$\begin{aligned} \frac{\partial q_1}{\partial p_1} &= \frac{q_1}{p_1 (r-1)} - \frac{\kappa r}{p_1 (r-1)} \frac{q_1}{\kappa p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}} \\ &\quad \frac{q_1}{p_1 (r-1)} \left[1 - \frac{\kappa r}{\kappa p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}} \right] \end{aligned} \quad (5)$$

The advantage of this form is that by multiplying Equation 5 by p_1/q_1 we easily get the elasticity

$$\eta_{11} = \frac{1}{r-1} \left[1 - \frac{\kappa r p_1^{\frac{r}{r-1}}}{\kappa p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}} \right] \quad (6)$$

– By similar derivations

$$\begin{aligned} \frac{\partial q_2}{\partial p_2} &= \frac{q_2}{p_2 (r-1)} \left[1 - \frac{r p_2^{\frac{r}{r-1}}}{\kappa p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}} \right] \\ \eta_{22} &= \frac{1}{r-1} \left[1 - \frac{r p_2^{\frac{r}{r-1}}}{\kappa p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}} \right] \end{aligned} \quad (7)$$

- As a starting point, a little Alchemy – I would like to generate a decent solution. In theory, I would like to talk about food (agricultural output) that is fairly inelastic. However, the CES function is not very stable at the point. So I will start by approximating the CES solution around

the point where $p_1 = 1.50$ and $p_2 = 1.25$. I would like to solve for the values of a and r such that the elasticity of demand for good 1 (food) is -1.05 and the elasticity of demand for the second good is -1.55. Mathematically, I solve

$$\min_{a,r} \left(\frac{1}{r-1} \left[1 - \frac{\left(\frac{1-a}{a}\right)^{\frac{1}{r-1}} r p_1^{\frac{r}{r-1}}}{\left(\frac{1-a}{a}\right)^{\frac{1}{r-1}} p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}} \right] + 1.05 \right)^2 + \left(\frac{1}{r-1} \left[1 - \frac{r p_2^{\frac{r}{r-1}}}{\left(\frac{1-a}{a}\right)^{\frac{1}{r-1}} p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}}} \right] + 1.25 \right)^2 \quad (8)$$

which yields $a \doteq 0.8274$ and $r \doteq 0.3750$. Assuming an income of \$ 100 and plugging these values into Equation 2 yields $q_1^*(1.50, 1.25) = 61.1111$ and $q_2^*(1.50, 1.25) = 6.6667$. Plugging these values into the Indirect Utility function in Equation 3 yields $v(p_1, p_2, y) = 46.4957$. To complete the circle, we use the Hicksian demand from Lecture 15

$$q_1^H(p_1, p_2, U^*) = \frac{\left(\frac{1-a}{a}\right)^{\frac{1}{r-1}} p_1^{\frac{1}{r-1}} U^*}{\left[a \left(\frac{1-a}{a}\right)^{\frac{r}{r-1}} p_1^{\frac{r}{r-1}} + p_2^{\frac{r}{r-1}} \right]^{\frac{1}{r}}} \quad (9)$$

where $q_1^H(1.5, 1.25, 46.4957) = 61.1111$ which is the same as the Marshallian demand.

- Next, consider what happens if we reduce the price of the first good to 1.40 – the Marshallian demand then becomes $q_1^*(1.40, 1.25, 100) = 65.6982$. I want to follow this through the discussion of compensating and equivalent variations.
 - The first step is to define indirect utility at the new price $v(1.40, 1.25, 100) = 49.5366$. As a “check” we compute the Hicksian demand for good 1 where $p_1 = 1.40$ and $U^* = 49.5366 \rightarrow q_1^H(.) = 65.6982$.

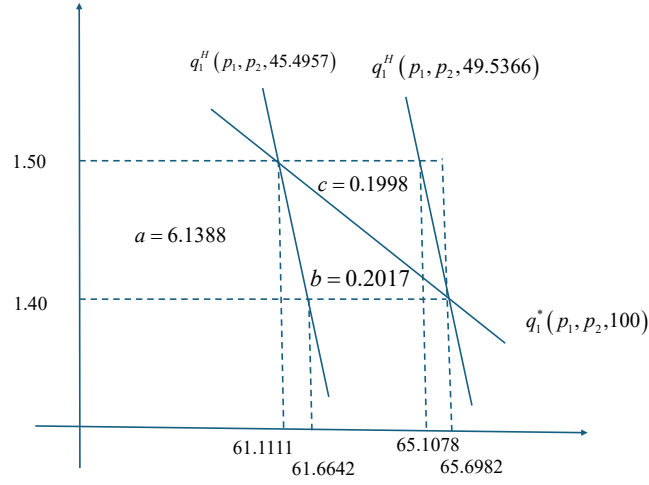


Figure 1: Willingness-To-Pay: Marshallian and Hicksian Measures

- Next, we can solve for the Hicksian demand for the first good at the new utility and the old price $q_1^H(1.50, 1.25, 49.5366) = 65.1078$.
- Going back to the original utility, the Hicksian demand at the original utility and the new price becomes $q_1^H(1.40, 1.25, 46.4957) = 61.6652$.