

Lecture XVI

Review of Dynamic Mathematics I

I. Introduction to the Dynamic Optimality

A. To introduce the problem of dynamic optimization, I would like to start with a discrete time problem. Specifically, we will use the example from Howitt “Some Microeconomics of Agricultural Resource Use.”

1. The basic problem is that y_t , the firm’s output at time t , is a function of variable inputs, x_t , and the level of capital stock, k_t . Further, capital stock can be augmented by investment, I_t . The firm’s overall problem can then be written as:

$$\max_{x, k, I} \sum_{t=0}^{T-1} \beta_t \pi(x_t, I_t, k_t) + \beta_T \pi(x_T, k_T)$$

$$k_{t+1} - k_t = \delta k_t + I_t$$

$$k_0 = \bar{k}$$

Within this formulation, the first equation embodies the objective function, the second equation is referred to as the equation of motion depicting how the capital changes over time, and the third equation is the initial condition on capital.

2. Rewriting this equation in Lagrangian form yields

$$L = \sum_{t=0}^{T-1} \left[\beta_t \pi(x_t, I_t, k_t) - \lambda_{t+1} (k_{t+1} - k_t - \delta k_t - I_t) \right] + \beta_T \pi(x_T, k_T)$$

This form of the problem yields three types of conditions.

- a. The first type of conditions are with respect to the control variables. This problem has two control variables, x_t and I_t which the decision maker can directly control. Specifically, the capital variable, k_t , is a state variable which can only be controlled through investment, I_t . The first of the control variables, x_t , has no long term effect. Thus, setting the level of x_t relies on similar rules as a static optimization problem:

$$\frac{\partial L}{\partial x_t} = \beta_t \frac{\partial \pi(\cdot)}{\partial x_t} = 0$$

$$\frac{\partial \pi(\cdot)}{\partial x_t} = 0$$

However, the derivative with respect to the investment variable, I_t , yields conditions dependent on λ_{t+1} . λ_{t+1} is the Lagrange multiplier on the equation of motion. As such, it depicts the marginal value of a change in the capital stock. In the jargon of dynamic optimization, this variable is referred to as the costate or adjoint variable. The second condition with respect to I_t is then:

$$\frac{\partial L}{\partial I_t} = \beta_t \frac{\partial \pi(\cdot)}{\partial I_t} + \lambda_{t+1} = 0$$

Thus, the decision regarding the optimal level of investment must be made considering the future value of the capital stock.

- b. The second condition is the partial with respect to λ_{t+1} , which guarantees that the equation of motion is satisfied.

$$\frac{\partial L}{\partial \lambda_{t+1}} = k_{t+1} - k_t - \delta k_t - I_t = 0$$

- c. Finally, taking the derivative with respect to the level of the state variable, capital stock yields:

$$\frac{\partial L}{\partial k_t} = \beta_t \frac{\partial \pi(\cdot)}{\partial k_t} - \lambda_t + (1 - \delta)\lambda_{t+1} = 0$$

Note that k_t appears in two equations of motion, one at time t and one at time $t+1$. Taking the derivative of the sum then yields two costate variables in the state equation. This equation can be rewritten as:

$$\lambda_{t+1} - \lambda_t = -\beta_t \frac{\partial \pi(\cdot)}{\partial k_t} - \lambda_{t+1} \delta$$

Note that this equation can be written as a difference equation

$$\Delta \lambda_{t+1} = -\beta_t \frac{\partial \pi(\cdot)}{\partial k_t} - \lambda_{t+1} \delta$$

Thus, the solution of this equation depicts how the costate variable, or the value of the capital constraint, changes over time.

- d. Taking some liberty with the mathematics, and letting Δt approach zero. The above equation becomes

$$\lambda'(t) = -\beta(t) \frac{\partial \pi(x(t), I(t), k(t))}{\partial k(t)} - \lambda(t) \delta = 0$$

Which is the continuous representation of the dynamic optimization problem.

3. This discussion then leads to two different types of dynamic mathematical problems. The first problem, given that time is specified in discrete increments is referred to as a difference equation. The second, given that time is continuous is referred to as a differential equation. In this course, we will emphasize the continuous time formulation.
4. Finally, the discrete formulation above can be used to demonstrate Pontryagin's optimality conditions. Specifically, the objective function at each increment in time can be written as a function called the Hamiltonian:

$$H_t(x_t, I_t, k_t, \lambda_t) = \beta_t \pi(x_t, I_t, k_t) - \lambda_{t+1} (-\delta k_t - I_t)$$

Remembering that the full equation of motion term in the original objective function was

$$\lambda_{t+1}(k_{t+1} - k_t - \delta k_t - I_t)$$

The optimal control problem can be rewritten as

$$L = \sum_{t=0}^{T-1} \left\{ H_t(x_t, I_t, k_t, \lambda_t) - \lambda_{t+1}(k_{t+1} - k_t) \right\} + \beta_T \pi(x_T, k_T)$$

This formulation yields the three classic Pontryagin conditions:

a. The Optimality/Hamiltonian Conditions

$$\frac{\partial H}{\partial u_t} = 0 \Rightarrow \begin{cases} \frac{\partial L}{\partial x_t} = \beta_t \frac{\partial \pi(\cdot)}{\partial x_t} = 0 \\ \frac{\partial L}{\partial I_t} = \beta_t \frac{\partial \pi(\cdot)}{\partial I_t} + \lambda_{t+1} = 0 \end{cases}$$

b. The Adjoint/Costate Conditions

$$\frac{\partial H}{\partial \lambda_{t+1}} = g(s_t, u_t) = s_{t+1} - s_t \Rightarrow -\delta k_t - I_t = k_{t+1} - k_t$$

c. The Equation of Motion Constraint

$$\frac{\partial H}{\partial s_t} = \lambda_{t+1} - \lambda_t \Rightarrow \beta_t \frac{\partial \pi(\cdot)}{\partial k_t} - \lambda_{t+1} \delta = \lambda_{t+1} - \lambda_t$$

II. Review of Ordinary Differential Equations

A. Thus, from the discussion of the optimal control framework above, we see that dynamic optimization in continuous time involves solving differential equations. This week, we will focus on the material presented in Chapter 2 of Tu.

B. Definitions

1. Ordinary differential equations (ODE) are equations involving total derivatives:

$$\varphi\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \dots, \frac{d^nx}{dt^n}, t\right) = 0$$

2. The order of an ODE is given by the highest order derivative in the equation. The degree of an ODE is given by the highest exponent of any derivative. For example

$$\ddot{x} + a \dot{x}^3 + b x + c = 0$$

is a second order ODE of degree 3.

3. An ODE is said to be linear if it is of the first degree in the dependent variable and its derivatives

$$a_0(t) \frac{d^nx}{dt^n} + a_1(t) \frac{d^{n-1}x}{dt^{n-1}} + \dots + a_{n-1}(t) \dot{x} + a_n(t) x = 0$$

4. An ODE containing terms of a higher degree (than the first) is called a nonlinear ODE

$$\ddot{x} + a \dot{x} + b x^2 = 0$$

5. The solution of an ODE is defined as any relationship between variables not involving the derivatives which satisfies the ODE. The solution is also called the integral curve.
6. Theorem 2.0 Let $x'(t) = f(x,t)$ be continuous with continuous derivatives $\partial f/\partial x$ for x in the domain D and t in the interval I , i.e. $x \in D$ and $t \in I$, there exists a solution $x^*(t)$ defined uniquely in some neighborhood of (x_0, t_0) satisfying the initial conditions $x^*(t_0) = x_0$.

C. First Order Linear Differential Equations

1. First order linear differential equations are equations of the form

$$a_0(t) \dot{x} + a_1(t) x = G(t)$$

The first order linear differential equation can be written as:

$$\dot{x} + a(t) x = g(t)$$

Given that $a_0(t)$ is not equal to zero.

- a. The first solution to be derived is the Homogeneous case where $g(t)$ is set equal to zero.

$$\dot{x} + a x = 0 \quad \text{given} \quad x(0) = x_0$$

To solve this problem, we rewrite the original problem as:

$$\frac{dx}{dt} = -a x$$

$$\frac{dx}{x} = -a dt$$

$$\int \frac{dx}{x} = - \int a dt$$

$$\ln(x) = -a t + c$$

$$x(t) = A e^{-at}$$

- b. The second scenario in the first order linear differential equation is the non homogeneous case. Under this scenario, the right hand side of the differential equation is not zero:

$$\dot{x} + ax = b, \quad x(0) = x_0$$

The procedure in Tu involves substituting $y = x - b/a$ for x which yields a homogeneous equation:

$$y = x - \frac{b}{a} \Rightarrow dy = dx$$

Thus the differential equation can be written as

$$\dot{y} + a \left(y + \frac{b}{a} \right) = b$$

$$\dot{y} + a y = 0$$

$$y(t) = y_0 e^{-at}$$

$$x(t) = \left(x_0 - \frac{b}{a} \right) e^{-at} + \frac{b}{a}$$

- c. This representation points to the division of the solution into two components. The first component is called the complementary function, which is the solution to the homogeneous part of the of the problem, $x_c(t) = A e^{-at}$. The other

part is the particular integral which is called the “equilibrium” portion of the solution, $x_p(t)=b/a$. The complete solution is then the sum of those two parts $x(t) = x_c(t) + x_p(t)$.

- d. To develop a fairly general form of the linear problem, let us start by assuming that

$$\dot{x} + ax = g(t)$$

Given that $a \neq 0$, this equation actually amounts to finding some function $G(t)$ such that

$$\frac{d}{dt}x + ax = \frac{d}{dt}(G(t))$$

Starting from the solution of the differential equation with constant coefficients, we have $x = c \text{Exp}[-a t]$. Reversing this expectation, $c = y \text{Exp}[a t]$. Hence

$$\frac{d}{dt}(x e^{at}) = 0$$

$$e^{at}(\dot{x} + ax) = 0$$

Thus

$$\dot{x} + ax = g(t)$$

$$e^{at}(\dot{x} + ax) = e^{at}g(t)$$

$$d(e^{at}x) = e^{at}g(t)$$

$$e^{at}x = \int_t e^{as}g(s)ds$$

$$x = e^{-at} \int_t e^{as}g(s)ds$$

For example

$$\dot{x} + 3x = t + e^{-2t}$$

$$d(x e^{3t}) = t e^{3t} + e^t$$

$$x = e^{-3t} \int_t (s e^{3s} + e^s) ds + c e^{-3t}$$

Integrating the first term by parts

$$\int u dv = uv - \int v du$$

$$d v = e^{3s} \Rightarrow v = \frac{1}{3} e^{3s}$$

$$u = s \Rightarrow d u = ds$$

$$\int s e^{3s} ds = \frac{t}{3} e^{3t} - \int \frac{1}{3} e^{3s} ds$$

$$= \frac{t}{3} e^{3t} - \frac{1}{9} e^{3t}$$

The solution then becomes

$$x = \frac{t}{3} - \frac{1}{9} + e^{-2t} + c e^{-3t}$$

- e. Following the same logic for
 $\dot{x} + p(t)x = g(t)$

we want to choose a function $\mu(t)$ such that

$$\begin{aligned}\mu(t)(\dot{x} + p(t)x) &= (\mu(t)x)' \\ &= \mu'(t)x + \mu(t)\dot{x}\end{aligned}$$

which implies that

$$d(\mu(t)x) = \mu(t)g(t)$$

$$\mu(t)x = \int_t \mu(s)g(s)ds$$

$$x = \frac{1}{\mu(t)} \int_t \mu(s)g(s)ds$$

The choice of $\mu(t)$ is then determined by

$$\mu(t)(\dot{x} + p(t)x) = \mu'(t)x + \mu(t)\dot{x}$$

$$\mu(t)p(t) = \mu'(t)$$

$$p(t) = \frac{\mu'(t)}{\mu(t)}$$

$$\int_t p(s)ds = \ln(\mu(t))$$

$$\mu(t) = \exp\left(\int_t p(s)ds\right)$$