

Lecture XV

Overview of Duality in Mathematical Programming

I. Derivation of the Dual Space

A. A simple linear problem

$$\max_{x, \lambda} c_1 x_1 + c_2 x_2 + c_3 x_3 + \lambda_1 (b_1 - a_{11} x_1 - a_{12} x_2 - a_{13} x_3) \\ \lambda_2 (b_2 - a_{21} x_1 - a_{22} x_2 - a_{23} x_3)$$

1. In this framework x_1 , x_2 , and x_3 are choice variables or actions such as levels of plantings.
2. λ_1 and λ_2 are shadow values or marginal values of each resource constraint.
3. c_1 , c_2 , and c_3 are objective value coefficients for each activity.
4. b_1 and b_2 are maximum resource levels (or right-hand sides).
5. The a_{ij} are resource use coefficients.

B. Distributing and recollecting

$$c_1 x_1 + c_2 x_2 + c_3 x_3 + \lambda_1 b_1 - \lambda_1 a_{11} x_1 - \lambda_1 a_{12} x_2 - \lambda_1 a_{13} x_3 + \\ \lambda_2 b_2 - \lambda_2 a_{21} x_1 - \lambda_2 a_{22} x_2 - \lambda_2 a_{23} x_3 \\ b_1 \lambda_1 + b_2 \lambda_2 + x_1 c_1 - x_1 a_{11} \lambda_1 - x_1 a_{21} \lambda_2 + x_2 c_2 - x_2 a_{12} \lambda_1 - x_2 a_{22} \lambda_2 + \\ x_3 c_3 - x_3 a_{13} \lambda_1 - x_3 a_{23} \lambda_2 \\ b_1 \lambda_1 + b_2 \lambda_2 + x_1 (c_1 - a_{11} \lambda_1 - a_{21} \lambda_2) + x_2 (c_2 - a_{12} \lambda_1 - a_{22} \lambda_2) + \\ x_3 (c_3 - a_{13} \lambda_1 - a_{23} \lambda_2)$$

1. The transformed formulation yields the same optimum as the first formulation, but it is a minimization problem.
2. The transformed problem minimizes the value of resources used (λ_i) in the solution.

C. Matrix formulation

$$L = c'x + \lambda'(b - Ax) \\ = c'x + \lambda'b - \lambda'Ax \\ = b'\lambda + x'c - x'A'\lambda \\ = b'\lambda + x'(c - A'\lambda)$$

II. A nonlinear perspective

A. Working from the nonlinear problem

$$L = f(x) - \lambda'(G(x)) \Rightarrow$$

$$L = f(x^*) + \nabla_x f(x^*)' x + \lambda'(G(x^*) - \nabla_x G(x^*)x) \\ = G(x^*)' \lambda + x' \left(\nabla_x f(x^*) - \nabla_x G(x^*)' \lambda \right) + f(x^*)$$