

Lecture XIV

Practical Optimization Problems and Computer Code

I. Unconstrained Optimization

A. *Precision Agriculture*: Using mathematica to solve optimization problems

B. *Estimation of the Inverse Hyperbolic Sine Transformation*: Using Gauss to optimize likelihood functions.

1. Most courses in statistical inference present the concept of maximum likelihood. Specifically, given a sample of data points, we can write the likelihood function of a set of parameters as:

$$L = \prod_{i=1}^n \left[\frac{1}{\sqrt{\sigma^2 2\pi}} \exp\left(-\frac{\{x_i - \mu\}^2}{2\sigma^2}\right) \right]$$

Taking the natural log of this expression yields

$$\ln(L) = \sum_{i=1}^n \left[-\frac{1}{2} \ln(2\pi) - \frac{1}{2} \ln(\sigma^2) - \frac{\{x_i - \mu\}^2}{2\sigma^2} \right]$$

Taking the derivative with respect to μ yields

$$\frac{\partial \ln(L)}{\partial \mu} = \sum_{i=1}^n \frac{x_i - \mu}{\sigma^2} = 0$$

$$n\mu = \sum_{i=1}^n x_i$$

$$\mu = \frac{1}{n} \sum_{i=1}^n x_i$$

Thus, we derive a maximum likelihood estimate of the mean. In this case the maximum likelihood is “closed form”.

2. Another potential likelihood function involves the inverse hyperbolic sine transformation to normality. This distribution allows variables to be both skewed and kurtotic. Ramirez, Moss and Boggess estimated corn, soybean and wheat yields over time using this transformation. The log-likelihood function for this distribution with a linear time trend can be written as:

$$\ln(L^*) = -\frac{n}{2} \ln(\sigma^2) - \frac{1}{2} \sum_{i=1}^n \frac{\{z_i - \mu\}^2}{\sigma^2} - \frac{1}{2} \sum_{i=1}^n (1 + z_i^2 \theta^2)$$

$$v_i = y_i - \alpha_0 - \alpha_1 i$$

$$z_i = \frac{\ln \left[v_i \theta + \sqrt{1 + v_i^2 \theta^2} \right]}{\theta}$$

The parameters to be estimated, or maximized with respect to are α_0 , α_1 , θ , μ , and σ . Guess what, there exist no closed form solution to those first order conditions.

3. Several numerical solvers are available for this problem, but I want to outline the solution in Gauss.

4. Gauss's optimization routine (for Gauss 3.x) is a subroutine called `optmum`. To use `optmum` you must
 - a. Load the `optmum` library
 - b. Create a subroutine which uses the parameters to be estimated to calculate the value of the objective function (log-likelihood function).
 - i. You may also want to specify the objective function gradients and the Hessian matrix.
 - ii. Note that `optmum` minimizes the specified objective function. Therefore, you want to specify negative of the likelihood function.
 - c. Initialize the algorithm
 - i. Initialize by specifying the starting values of the choice variables.
 - ii. Initialize by choosing the an optimization algorithm.
 - d. Call the optimization function.
5. *An example:* Instead of focusing on the more complex example, I want to discuss the estimation of the mean and variance.

```
library optmum;
#include optmum.ext;
opset;

load x[20,1]=indta.dta;

proc ml(b);
  local err;
  err=x-b[1];
  retp( rows(err)*ln(b[2])+sumc((x-b[1]).^2./b[2]) );
endp;

b0={10,1.5};
{fhat,bhat,ghat,retcode}=optprt(optmum(&ml,b0));
```

Some notes

- a. First note `b` is a gauss vector with `b[1]` containing the value of the mean and `b[2]` containing the value of the variance.
- b. Second, note that the default algorithm is BFGS with numerical gradients.
- c. Third, note that estimating the variance instead of the standard deviation raises some potential problems.
Why?

II. Linear Equality Constraints in Optimization

A. *Designing an Optimal Portfolio*: Optimization using MINOS or GAMS

GAMS 2.25.051 386/486 DOS 01/07/88 21:59:58 PAGE 1
SIMPLE PORTFOLIO MODEL (PORT,SEQ=50)

```

2 * THIS SIMPLE PORTFOLIO SELECTION MODEL EXAMINES
3 * INVESTMENT ALTERNATIVES IN THE BOND MARKET. THE SELECTION
4 * IS CONSTRAINED BY RATING AND MATURITY CONSIDERATIONS.
5 *
6 * REFERENCE: CONTROL DATA CORPORATION, IFPS/OPTIMIM - USERS MANUAL,
7 * MINNEAPOLIS, 1984.
8
9 SETS B BONDS / MUNICIPAL-A, MUNICIPAL-B, CORPORATE, US-SER-E,
US-SER-F /
10 G(B) GROUPING / CORPORATE, US-SER-E, US-SER-F /
11
12 TABLE YDAT(B,*) YIELD DATA
13
14 RATING MATURITY YIELD TAX-RATE
15 MUNICIPAL-A 2 9 4.3
16 MUNICIPAL-B 5 2 4.5
17 CORPORATE 2 15 5.4 .5
18 US-SER-E 1 4 5.0 .5
19 US-SER-F 1 3 4.4 .5
20
21 VARIABLES INVESTMENT(B)
22 TINVEST TOTAL INVESTMENT
23 RETURN
24 POSITIVE VARIABLE INVESTMENT
25
26 EQUATIONS GROUPMIN MINIMUM INVESTMENT IN GROUP G
27 RDEF RATING DEFINITION
28 MDEF MATURITY DEFINITION
29 IDEF TOTAL RETURN DEFINITION
30 TDEF TOTAL INVESTMENT DEFINITION ;
31
32 TINVEST.UP = 10;
33
34 GROUPMIN.. SUM(G, INVESTMENT(G)) =G= 4;
35
36 RDEF.. SUM(B, YDAT(B,"RATING ")*INVESTMENT(B)) =L= 1.4*TINVEST;
37
38 MDEF.. SUM(B, YDAT(B,"MATURITY")*INVESTMENT(B)) =L= 5.0*TINVEST;
39
40 TDEF.. TINVEST =E= SUM(B, INVESTMENT(B));
41
42 IDEF.. RETURN =E= SUM(B, YDAT(B,"YIELD")/100*(1-YDAT(B,"TAX-RATE"))
*INVESTMENT(B));
43
44 MODEL PORT / ALL / ; SOLVE PORT MAXIMIZING RETURN USING LP;

```

GAMS 2.25.051 386/486 DOS 01/07/88 21:59:58 PAGE 2
SIMPLE PORTFOLIO MODEL (PORT,SEQ=50)
Symbol Listing

SYMBOL TYPE REFERENCES

B	SET	DECLARED	9	DEFINED	9	REF	10	
		12 21 2*34 2*35 36 3*37						
		CONTROL	34	35	36	37		
G	SET	DECLARED	10	DEFINED	10	REF	33	
		CONTROL	33					
GROUPMIN	EQU	DECLARED	26	DEFINED	33	IMPL-ASN	38	
		REF	38					
IDEF	EQU	DECLARED	29	DEFINED	37	IMPL-ASN	38	
		REF	38					
INVESTMENT	VAR	DECLARED	21	24	IMPL-ASN	38		
		REF	33	34 35 36 37				
MDEF	EQU	DECLARED	28	DEFINED	35	IMPL-ASN	38	
		REF	38					
PORT	MODEL	DECLARED	38	DEFINED	38	IMPL-ASN	38	
		REF	38					
RDEF	EQU	DECLARED	27	DEFINED	34	IMPL-ASN	38	
		REF	38					
RETURN	VAR	DECLARED	23	IMPL-ASN	38	REF	37	
		38						
TDEF	EQU	DECLARED	30	DEFINED	36	IMPL-ASN	38	
		REF	38					
TINVEST	VAR	DECLARED	22	IMPL-ASN	38	ASSIGNED	32	
		REF	34 35 36					
YDAT	PARAM	DECLARED	12	DEFINED	12	REF	34	
		35 2*37						

SETS

B BONDS
G GROUPING

PARAMETERS

YDAT YIELD DATA

VARIABLES

INVESTMENT
RETURN
TINVEST TOTAL INVESTMENT

EQUATIONS

GROUPMIN MINIMUM INVESTMENT IN GROUP G
IDEF TOTAL RETURN DEFINITION

MDEF MATURITY DEFINITION
RDEF RATING DEFINITION

GAMS 2.25.051 386/486 DOS 01/07/88 21:59:58 PAGE 3
SIMPLE PORTFOLIO MODEL (PORT,SEQ=50)
Symbol Listing

EQUATIONS

TDEF TOTAL INVESTMENT DEFINITION

MODELS

PORT

COMPILATION TIME = 0.160 SECONDS VERID MW2-00-051

GAMS 2.25.051 386/486 DOS 01/07/88 21:59:58 PAGE 4
SIMPLE PORTFOLIO MODEL (PORT,SEQ=50)
Equation Listing SOLVE PORT USING LP FROM LINE 38

---- GROUPMIN =G= MINIMUM INVESTMENT IN GROUP G

GROUPMIN.. INVESTMENT(CORPORATE) + INVESTMENT(US-SER-E)
+ INVESTMENT(US-SER-F) =G= 4 ; (LHS = 0 ***)

---- RDEF =L= RATING DEFINITION

RDEF.. 2*INVESTMENT(MUNICIP-A) + 5*INVESTMENT(MUNICIP-B)
+ 2*INVESTMENT(CORPORATE) + INVESTMENT(US-SER-E) + INVESTMENT(US-SER-F)
- 1.4*TINVEST =L= 0 ; (LHS = 0)

---- MDEF =L= MATURITY DEFINITION

MDEF.. 9*INVESTMENT(MUNICIP-A) + 2*INVESTMENT(MUNICIP-B)
+ 15*INVESTMENT(CORPORATE) + 4*INVESTMENT(US-SER-E)
+ 3*INVESTMENT(US-SER-F) - 5*TINVEST =L= 0 ; (LHS = 0)

---- IDEF =E= TOTAL RETURN DEFINITION

IDEF.. - 0.043*INVESTMENT(MUNICIP-A) - 0.045*INVESTMENT(MUNICIP-B)
- 0.027*INVESTMENT(CORPORATE) - 0.025*INVESTMENT(US-SER-E)
- 0.022*INVESTMENT(US-SER-F) + RETURN =E= 0 ; (LHS = 0)

---- TDEF =E= TOTAL INVESTMENT DEFINITION

TDEF.. - INVESTMENT(MUNICIP-A) - INVESTMENT(MUNICIP-B)
- INVESTMENT(CORPORATE) - INVESTMENT(US-SER-E) - INVESTMENT(US-SER-F)
+ TINVEST =E= 0 ; (LHS = 0)

GAMS 2.25.051 386/486 DOS 01/07/88 21:59:58 PAGE 5
SIMPLE PORTFOLIO MODEL (PORT,SEQ=50)
Column Listing SOLVE PORT USING LP FROM LINE 38

---- INVESTMENT

INVESTMENT(MUNICIP-A)
 (.LO, .L, .UP = 0, 0, +INF)
 2 RDEF
 9 MDEF
-0.043 IDEF
-1 TDEF

INVESTMENT(MUNICIP-B)
 (.LO, .L, .UP = 0, 0, +INF)
 5 RDEF
 2 MDEF
-0.045 IDEF
-1 TDEF

INVESTMENT(CORPORATE)
 (.LO, .L, .UP = 0, 0, +INF)
 1 GROUPMIN
 2 RDEF
 15 MDEF
-0.027 IDEF
-1 TDEF

REMAINING 2 ENTRIES SKIPPED

---- TINVEST TOTAL INVESTMENT

TINVEST
 (.LO, .L, .UP = -INF, 0, 10)
-1.4 RDEF
-5 MDEF
1 TDEF

---- RETURN

RETURN
 (.LO, .L, .UP = -INF, 0, +INF)
1 IDEF

GAMS 2.25.051 386/486 DOS 01/07/88 21:59:58 PAGE 6
SIMPLE PORTFOLIO MODEL (PORT,SEQ=50)
Model Statistics SOLVE PORT USING LP FROM LINE 38

MODEL STATISTICS

BLOCKS OF EQUATIONS	5	SINGLE EQUATIONS	5
BLOCKS OF VARIABLES	3	SINGLE VARIABLES	7
NON ZERO ELEMENTS	27		

GENERATION TIME = 0.330 SECONDS

EXECUTION TIME = 0.440 SECONDS VERID MW2-00-051

GAMS 2.25.051 386/486 DOS 01/07/88 21:59:58 PAGE 7
 SIMPLE PORTFOLIO MODEL (PORT,SEQ=50)
 Solution Report SOLVE PORT USING LP FROM LINE 38

S O L V E S U M M A R Y

MODEL PORT OBJECTIVE RETURN
 TYPE LP DIRECTION MAXIMIZE
 SOLVER MINOS5 FROM LINE 38

**** SOLVER STATUS 1 NORMAL COMPLETION
 **** MODEL STATUS 1 OPTIMAL
 **** OBJECTIVE VALUE 0.2984

RESOURCE USAGE, LIMIT 0.600 1000.000
 ITERATION COUNT, LIMIT 5 1000

M I N O S 5.3 (Nov 1990) Ver: 225-386-02
 = = = = =

B. A. Murtagh, University of New South Wales
 and
 P. E. Gill, W. Murray, M. A. Saunders and M. H. Wright
 Systems Optimization Laboratory, Stanford University.

Work space allocated -- 0.04 Mb

EXIT -- OPTIMAL SOLUTION FOUND

	LOWER	LEVEL	UPPER	MARGINAL
--- EQU GROUPMIN	4.000	7.364	+INF	.
--- EQU RDEF	-INF	.	.	0.006
--- EQU MDEF	-INF	.	.	0.002
--- EQU IDEF	.	.	.	1.000
--- EQU TDEF	.	.	.	-0.009

GROUPMIN MINIMUM INVESTMENT IN GROUP G
 RDEF RATING DEFINITION
 MDEF MATURITY DEFINITION
 IDEF TOTAL RETURN DEFINITION
 TDEF TOTAL INVESTMENT DEFINITION

--- VAR INVESTMENT

	LOWER	LEVEL	UPPER	MARGINAL
MUNICIP-A	.	2.182	+INF	.
MUNICIP-B	.	0.455	+INF	.
CORPORATE	.	.	+INF	-0.030
US-SER-E	.	7.364	+INF	.

US-SER-F . . +INF -6.364E-4

GAMS 2.25.051 386/486 DOS 01/07/88 21:59:58 PAGE 8
 SIMPLE PORTFOLIO MODEL (PORT,SEQ=50)
 Solution Report SOLVE PORT USING LP FROM LINE 38

	LOWER	LEVEL	UPPER	MARGINAL
--- VAR TINVEST	-INF	10.000	10.000	0.030
--- VAR RETURN	-INF	0.298	+INF	.

TINVEST TOTAL INVESTMENT
 RETURN

**** REPORT SUMMARY : 0 NONOPT
 0 INFEASIBLE
 0 UNBOUNDED

EXECUTION TIME = 0.110 SECONDS VERID MW2-00-051

USER: Food and Resource Economics S920916-1049Ax-MW2
 University of Florida, Gainesville

**** FILE SUMMARY

INPUT C:\GAMS\MODLIB\PORT.50
 OUTPUT C:\GAMS\PORT.LST

III. Linear Inequality Constraints in Optimization

A. Freund's Portfolio Model: An application of MINOS

- Starting with the basic model of portfolio choice:

$$\min_x x' \Omega x$$

$$st \quad x' \mu > Y^*$$

Freund showed that the expected utility of a normally distributed gamble given negative exponential preferences could be written as

$$E[U[x]] = \mu(x) - \frac{\rho}{2} \sigma^2(x)$$

Extending this formulation to vector form, the problem can be expressed as:

$$\max_x x' \mu - \frac{\rho}{2} x' \Omega x.$$

- Given this formulation, Freund constructs a table of resource requirements for the farm level planning problem:

	Early Irish Potatoes	Corn	Beef	Fall Cabbage	Available
		Land			
January-June	1.199	1.382	2.776	.000	60.0
July-December	.000	1.382	2.776	.482	60.0
		Production Capital			
Period 1	1.064	.484	.038	.000	24.0
Period 2	-2.064	.020	.107	.229	12.0
Period 3	-2.064	-1.504	-1.145	--1.229	0.0
		Managerial Labor			
Period 1	5.276	4.836	.000	.000	799.0
Period 2	2.158	4.561	.000	4.198	867.0
Period 3	.000	4.561	.000	13.606	783.0
		Unit Level of \$100.0			
	1.000	1.000	1.000	1.000	

3. The variance matrix is defined as

$$\Omega = \begin{pmatrix} 7304.69 & 903.89 & -688.73 & -182.05 \\ & 620.16 & -471.14 & 110.43 \\ & & 1124.64 & 750.69 \\ & & & 3689.53 \end{pmatrix}$$

4. Letting x_1 be Irish potatoes, x_2 be corn, x_3 be beef, and x_4 be fall cabbage, the optimization problem is written as:

$$\max_x \left(1 \quad 1 \quad 1 \quad 1 \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} - \frac{\rho}{2} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' \Omega \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

subject to

$$\begin{aligned} 1.199x_1 + 1.382x_2 + 2.776x_3 + 0x_4 &\leq 60.0 \\ 0x_1 + 1.382x_2 + 2.776x_3 + .482x_4 &\leq 60.0 \\ 1.064x_1 + .484x_2 + .038x_3 + 0x_4 &\leq 24.0 \\ -2.064x_1 + .020x_2 + .107x_3 + .229x_4 &\leq 12.0 \\ -2.046x_1 - 1.504x_2 - 1.145x_3 - 1.229x_4 &\leq 0.0 \\ 5.276x_1 + 4.836x_2 + 0x_3 + 0x_4 &\leq 799.0 \\ 2.158x_1 + 4.561x_2 + 0x_3 + 4.198x_4 &\leq 867.0 \\ 0x_1 + 4.146x_2 + 0x_3 + 13.607x_4 &\leq 787.0 \end{aligned}$$

5. Coding in MINOS

- MINOS is a collection of Fortran subroutines created to solve large, nonlinear optimization problems. User input to MINOS comes through two mechanisms

- i. The SPECS file which gives MINOS information such as the number of variables, rows, nonlinear variables, etc. on the problem to be solved.
 - ii. Modified Fortran subroutines. There are three subroutines which are typically modified: Funobj which returns the nonlinear component of the objective function, Funcon which returns the nonlinear component of the constraint matrix, and Matmod which can be used to solve a sequence of related problems.
- b. Given the above specification of the objective function we are interested in creating a nonlinear objective function component.

The first step is to write a calling routine which will call the main body of MINOS code.

```

PARAMETER(NWCORE=30000)
DOUBLE PRECISION Z(NWCORE)
COMMON /ZCOMM / Z
C
CALL MINOS1( Z,NWCORE )
STOP
C
C End of MAIN
END

After the calling routine we write the actual FUNOBJ subroutine
SUBROUTINE FUNOBJ( MODE, N, X, CE, G, NSTATE, NPROB, Z, NWCORE )
IMPLICIT REAL*8(A-H,O-Z)
DOUBLE PRECISION X(N), G(N), Z(NWCORE),ROMEGA(4,4),VAR
DATA ROMEGA/ 7304.69, 903.89,-688.73,-1862.05,
1 903.89, 620.16,-471.14, 110.43,
2 -688.73,-471.14,1124.64, 750.69,
3 -1862.05, 110.43, 750.69, 3689.53/
C
C VARIANCE DEFINED
C
RISKA=1.0/1250.
IF(NSTATE.EQ.1) THEN
DO 100 I=1,4
WRITE(IPRINT,1000) (ROMEGA(I,J),J=1,4)
100 CONTINUE
ENDIF
CE=0.0
DO 200 I=1,4
DO 150 J=1,4
CE=CE-RISKA/2.0*X(I)*X(J)*ROMEGA(I,J)
150 CONTINUE
200 CONTINUE
DO 300 I=1,4
G(I)=0.0
DO 250 J=1,4
G(I)=G(I)-RISKA*X(J)*ROMEGA(I,J)
250 CONTINUE
300 CONTINUE
1000 FORMAT(' ',10F10.2)
    
```

Next, on most operating systems you will want to append
 “dummy” Funcon and Matmod routines.

c. The SPECS file

```
BEGIN FREUND
  MINIMIZE
  LIST          200
  OBJECTIVE      VARIANCE
  NONLINEAR OBJECTIVE VARIABLES 4
  RHS            RHS01
  BOUNDS         BOUND01
  VERIFY GRADIENTS
  SOLUTION FILE   20
END FREUND
NAME      FREUND
ROWS
L LAND1
L LAND2
L CAPTL1
L CAPTL2
L CAPTL3
L MANGT1
L MANGT2
L MANGT3
N CERTAIN
COLUMNS
POTATO LAND1  1.199    CAPTL1    1.064
POTATO CAPTL2 -2.064    CAPTL3    -2.064
POTATO MANGT1  5.276    MANGT2     2.158
POTATO CERTAIN 1.000
CORN   LAND1  1.382    LAND2     1.382
CORN   CAPTL1  .484    CAPTL2     .020
CORN   CAPTL3 -1.504    MANGT1     4.836
CORN   MANGT2  4.561    MANGT3     4.146
CONR   CERTAIN 1.000
BEEF   LAND1  2.776    LAND2     2.776
BEEF   CAPTL1  .038    CAPTL2     .107
BEEF   CAPTL3 -1.145    CERTAIN    1.000
CABBAG LAND2  .482    CAPTL2     .229
CABBAG CAPTL3 -1.229    MANGT2     4.198
CABBAG MANGT3 13.606    CERTAIN    1.000
RHS
RHS01  LAND1  60.000    LAND2     60.000
RHS01  CAPTL1 24.000    CAPTL2    12.000
RHS01  MANGT1 799.00    MANGT2    867.00
RHS01  MANGT3 783.00
ENDRUN
```

A. *Spatial Equilibrium Models*: An application of GAMS

B. *Direct Utility Maximization Models*: An application of MINOS

IV. Nonlinear Constrained Models

A. *Chance Constrained Programming*: An application of MINOS

1. Profit maximization subject to a chance constraint
2. Cost minimization of a feed ration subject to a probabilistic protein requirement.

B. *Estimation of an ARCH Model*: Another application of Gauss or MINOS