

Lecture III

I. An in depth discussion of the uses of optimization techniques in agricultural and resource economics: The case of precision agriculture.

1. Introduction

One of the current rages in commercial agriculture is the evolution of precision agriculture. Several equipment companies are currently producing equipment such as combines, fertilizer spreaders, and sprayers to either provide input or respond to global positioning systems. The notion is that instead of treating fields as 100 acre units, new technologies can increase productivity by dividing the field into 30 foot by 30 foot unites. Fertilizer and pesticide applications can then be tailored to these small tracts.

The whole philosophy of precision agriculture would appear sound. Specifically, the use of information to equate the marginal cost and revenue for each productive unit would appear to increase economic efficiency. However, as is typically the case, the information is not costless. The new measuring devices for combines may cost several thousand dollars per machine. In addition, the farmer faces a substantial commitment in time and must acquire additional human capital. Thus, some study of the marginal costs and potential returns to this technology would appear appropriate.

This study examines the potential payoff of precision agriculture using a stylized primal approach. Specifically, we assume that each field is actually a combination of two fields with different primal production functions. The gain to precision agriculture is then the profit from identifying each tract of land. This gain must be compared against the cost of the new equipment. To simplify the study, we assume a single fertilizer input.

2. Production Functions Based on the Hyperbolic Tangent

The primal approach requires the assumption of a production function. In this study, we have decided to use the hyperbolic tangent function. Most production functions have at least one draw back. The traditional Cobb-Douglas function possesses a constant concavity ruling out increasing yields at an increasing rate. However, the hyperbolic tangent production function also has a problem in that it does not possess a stage III.

Graphically, the hyperbolic tangent function in its original form is presented in figure 1. Thus, we must modify it to make a legitimate production function. In this study, we define corn production as

$$f(x, \alpha_0, \alpha_1, \alpha_{\max}) = \frac{\alpha_{\max}}{2} [1 + \tanh(\alpha_0 + \alpha_1 x)] \quad (1)$$

where x represents the level of fertilizer applied while the parameters α_0 , α_1 , and $\alpha_{\max}$ allow us to control the shape of the function. The α_0 and α_1 parameters allow us to

change the location of the inflection point and the relative steepness of the production function at every point while $\alpha_{\max}$ sets the maximum yield. Letting $\alpha_0 = -1.25$, $\alpha_1 = .05$, and $\alpha_{\max} = 150$ yields the stylistic production function in figure 2.

Given this basic production function, we have several possibilities for varying production within the same field. For example, the soil may be of the same type, but past production practices may have changed the carryover level of nitrogen in the soil. In this case, it would be possible to envision several parallel production surfaces as depicted in figure 3. This figure depicts the difference between $\alpha_0 = -1.25$ and $\alpha_0 = -1.50$. The production function with $\alpha_0 = -1.50$ would represent those pieces of ground with a higher nitrogen carryover.

An alternative is the case where the soil types are fundamentally different. For example, a field may have sand and sandy loam. In this scenario, the productive capacity of the land as well as the relative nitrogen carryover may be altered as depicted in figure 4. In this scenario, we have changed from $\alpha_0 = -1.25$ to $\alpha_0 = -1.50$ and set $\alpha_{\max}$ to 125 for the poor soil and 175 for the good soil. Intuitively, this keeps the same average maximum yield at 150 bushels per acre.

3. Profit Maximization

Given these production relationships, we turn to the question of: What is precision agriculture worth? We will accomplish this analysis in stages. The concept of a technology which separates a large field into several subfields automatically raises the question of accuracy. The first analysis will assume that the new technology separates the fields perfectly. Afterwards, we will build on those results by allowing for slippage so that the high carryover subfield is treated as a low carryover subfield. This scenario may be visualized as either a inaccuracy of the technology or the limit of the unit of measure (i.e. 30 foot by 30 foot instead of 10 foot by 10 foot).

Perfect Separation

As a first step, we begin with the assumption that both the precision technology and the associated measurement are perfect. Assuming that δ is the percent of the land of the higher carryover type, the perfectly separated optimum is derived from the profit maximization problem:

$$\max_{x_1, x_2} \pi(\delta) = 3.00[\delta f(x_1, -1.25, .05, 150) + (1 - \delta) f(x_2, -1.50, .05, 150)] - .50(\delta x_1 + (1 - \delta) x_2) \quad (2)$$

where we have assumed that the corn price is \$3.00/bushel with a fertilizer price of \$.50/pound. The numeric optimum for this optimization problem indicates that $x_1^* = 69.71$ pounds of fertilizer while $x_2^* = 74.77$ pounds. The profit per acre for this allocation is \$408.81.

The profit without separation can be derived from the maximization problem:

$$\max_x \pi(\delta) = 3.00[\delta f(x, -1.25, .05, .150) + (1 - \delta) f(x, -1.50, .05, .150)] - .50x \quad (3)$$

where x is the level of fertilizer applied to both the high and low carryover parcels. The optimum level of fertilizer without separation can be derived as 72.57 pounds per acre. As we would expect, this level of fertilizer is about half way between the separated levels. The optimal level of profit without separation is \$408.66/acre. Thus, the net return to precision agriculture would be \$.15/acre or, assuming that the farmer farms two sections, \$193.63/year. The amortized value of this gain over 10 years at a discount rate of 8% is \$1299.28.

Graphically, the potential gain to precision agriculture is a function of the distribution between the two types of land. Figure 5 depicts the amortized value of precision farming as δ increases. In general, we see that the value of precision agriculture increases as δ approaches .5 and then declines. In addition, figure 5 also depicts the effect of a reduced dispersion between the production functions. Specifically, if we compare α_0 equal to -1.30 or -1.50 as opposed to -1.25 or -1.50, we see that the relative gain declines.

II. A Review of Expansions

First a review of the general notions of the Taylor expansion. The general notion of the Taylor expansion is the value of a function at a point, z , can be derived based on the value of that function at another point, a , the derivatives of the function evaluated at a , and the difference between a and z . Specifically,

$$f(z) = f(a) + f'(a)(z - a) + \frac{f''(a)}{2}(z - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(z - a)^n + \dots \quad (1)$$

and

$$f(z) = f(a) + f'(a)(z - a) + \frac{f''(a)}{2}(z - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(z - a)^n + R_{n+q} \quad (2)$$

where

$$R_{n+1} = \frac{f^{(n+1)}(a)}{(n+1)!}(z - z^*)^{n+1} \text{ for some } z^* \in [z, a].$$

Equation (1) states that the infinite order expansion is an equivalent function to the original function while equation (2) depicts the approximation over the first n derivatives. This approximation has an error, but the error is bounded based on the $n+1$ derivative.

The first notion is reasonable and somewhat intuitive in the case of the quadratic function:

$$f(x) = x^2 + ax + c.$$

Using the Taylor apparatus, we approximate the value of the function at x_1 given the value of the function at x_0 :

$$\begin{aligned} f(x_1) &= f(x_0) + (2x_0 + a)(x_1 - x_0) + \frac{2}{2}(x_1 - x_0)^2 \\ &= \{x_0^2 + ax_0 + c\} + \{2x_0x_1 + ax_1 - 2x_0^2 - ax_0\} + \{x_1^2 - 2x_1x_0 + x_0^2\} \\ &= x_1^2 + ax_1 + c \end{aligned}$$

so in the case of the quadratic function, the second order Taylor series expansion is exact. In order to induce an error term, consider the second order expansion of a cubic function:

$$\begin{aligned} g(x) &= x^3 + ax^2 + bx + c \\ g(x_1) &= g(x_0) + (3x_0^2 + 2ax_0 + b)(x_1 - x_0) + \frac{(6x_0 + 2a)}{2}(x_1 - x_0)^2 \end{aligned}$$

Other useful expansions include the Fourier expansion which depicts a function as the sum of sinusoids:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos(w_n t) + b_n \sin(w_n t)]$$

where w_n is a specific frequency. The notion is that a sufficient number of sinusoids can be added together to sufficiently proxy any general real function.

III. Optimality Conditions

The real workhorse in our discussion of optimality is the Taylor series expansion. First, some terminology. Let us begin by assuming that there exists a univariate function $f(x)$ defined over a given range $x \in [x_0, x_1]$. x^* is defined as a strong local minimum if $f(x^*) < f(x)$ for all $x \in N(x^*, \delta)$, $x \neq x^*$. x^* is a weak local minimum if $f(x^*) \leq f(x)$ for all $x \in N(x^*, \delta)$, $x \neq x^*$. However, local optimals are not sufficient, i.e. additional assumptions are required to guarantee that local optimals are also global.

Optimality from Taylor series: Note that any point $x \in N(x^*, \delta)$ can be derived as a Taylor series expansion of $f(x^*)$:

$$f(x) = f(x^*) + f'(x^*)(x - x^*) + \frac{1}{2}f''(x^*)(x - x^*)^2 + R_3.$$

Assuming that $R_3 \rightarrow 0$ as $\delta \rightarrow 0$ and rearranging we have

$$f(x) - f(x^*) = f'(x^*)(x - x^*) + \frac{1}{2} f''(x^*)(x - x^*)^2$$

If $f(x^*) < f(x)$ for all $x \in N(x^*)$ then $f(x) - f(x^*) > 0$ for $x \in N(x^*)$. Thus, if x^* is a strong local minimum then

$$f'(x^*)(x - x^*) + \frac{1}{2} f''(x^*)(x - x^*)^2 > 0.$$

Given that $x \in N(x^*)$ is a two sided concept $(x - x^*) \gtrless 0$. Thus, as $x \rightarrow x^*$ from the left,

$f'(x) < 0$. Similarly, if $x \rightarrow x^*$ from the right $f'(x) > 0$. If the first derivative is smooth, these conditions imply that $f'(x^*) = 0$ in order for the derivative to switch from negative to positive at x^* . In addition, if the first derivative is smooth so that the derivative switches from negative to positive, $f''(x^*) > 0$ for $x \in N(x^*)$.

IV. Weierstrass Theorem

The Weierstrass theorem guarantees that a continuous function has a maximum and a minimum on a closed and bounded interval.

1. A continuous function on a closed and bound interval. The notion of a continuous function is that there exists a function defined on an interval A such that

$$f: A \rightarrow \mathbb{R}^1 \text{ where } A = [x_0, x_1]$$

and

$$\lim_{x \rightarrow x^*} f(x) = \lim_{x \rightarrow x^*} f(x) = f(x^*)$$

for all $x^* \in A$. Thus, there are no holes in the functional mapping.

2. Given this general structure we know that the maximum or minimum must either be at a local maximum or minimum, or at a boundary point. The notion that the function is continuous presents the possibility of discontinuous derivatives. Thus, we are left with two general scenarios.
 - a. A continuous function on a closed and bounded interval does not possess a local maximum or minimum. Thus, either
 - i.) $f(x^*) - f(x) < 0$ for all $x^* > x$ in the interval in which case the function is constantly decreasing over the entire interval. Thus, the function

reaches its maximum at the lower bound of the interval and its minimum at the upper bound of the interval.

- ii.) $f(x^*) - f(x) > 0$ for all $x^* > x$ in the interval in which case the function is constantly increasing over the entire interval. Thus, the function reaches its maximum at the upper bound of the interval and its minimum at the lower bound of the interval.
- b. If the function possesses local maxima or minima in the interval the global maximum or minimum may occur at either one of those local optima or at the boundaries. Note that we only guarantee the existence of the maximum or minimum values, not the uniqueness of x associated with those values.
- c. *Regularity Conditions:* The smoothness of the calculus then allows for a more unique characterization. If $f(x) \in C^1$ then the relative minimum or maximum only occurs where $f'(x) = 0$. However, again, this does not guarantee a unique solution for x . Further, $f(x) \in C^2$ also does not guarantee a unique solution. However, if $f(x) \in C^2$ and $f''(x) < 0$ over the entire interval for a maximum or $f''(x) > 0$ over the entire interval for a minimum then the optima will be unique.