

Lecture XVII: Choosing the Firm's Input and Output Level

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July 25, 2014

Readings

Two Specifications of the Firm's Problems

Readings

- ▶ Friedman & Friedman Chapter 6

Two Specifications of the Firm's Problems

- ▶ The firm's problem can be specified in two related ways.
- ▶ First, we could ask what level of outputs minimize the cost of producing any fixed level of output

$$\begin{aligned} \min_{x_1, x_2} w_1 x_1 + w_2 x_2 \\ \text{s.t. } f(x_1, x_2) = Y \end{aligned} \quad (1)$$

- ▶ where x_1 and x_2 are input levels, w_1 and w_2 are prices for each input, and $Y = f(x_1, x_2)$ is a production function.
- ▶ The second specification involves selecting the level of inputs and output that maximize profit

$$\max_{x_1, x_2} p_Y f(x_1, x_2) - w_1 x_1 - w_2 x_2. \quad (2)$$

- ▶ As a starting point, we consider the first problem.
- ▶ At a risk of some complication, let us start with a two-input quadratic production function problem

$$\begin{aligned} & \min_{x_1, x_2} w_1 x_1 + w_2 x_2 \\ \text{s.t. } & f(x_1, x_2) = a_1 x_1 + a_2 x_2 + \frac{1}{2} (A_{11} x_1^2 + A_{12} x_1 x_2 + A_{22} x_2^2). \end{aligned} \quad (3)$$

- ▶ Recalling our discussion of the Lagrangian, I set up the constrained problem

$$L = w_1 x_1 + w_2 x_2 + \lambda \left(Y - a_1 x_1 - a_2 x_2 - \frac{1}{2} (A_{11} x_1^2 + A_{12} x_1 x_2 + A_{22} x_2^2) \right) \quad (4)$$

- ▶ Backing up for a moment, consider the two specifications of the consumer's problem that we worked with to develop the demand model.
- ▶ The first model maximized utility while holding income constant

$$\begin{aligned} \max_{q_1, q_2} U(q_1, q_2) \\ \text{s.t. } p_1 q_1 + p_2 q_2 \leq Y \end{aligned} \quad (5)$$

while the second one examined the choice of goods that minimized cost holding the utility constant

$$\begin{aligned} \min_{q_1, q_2} p_1 q_1 + p_2 q_2 \\ \text{s.t. } U(q_1, q_2) \geq U^* \end{aligned} \quad (6)$$

- ▶ Both of these solutions point to the same solution for one isoquant (i.e., if the minimum level of income in Equation 6 is used as the budget constraint in Equation 5 the solutions for q_1 and q_2 are identical).
- ▶ However, the Lagrange multipliers for each problem have a different meaning. In the case of Equation 6

$$L = p_1 q_1 + p_2 q_2 + \mu (U^* - U(q_1, q_2)) \quad (7)$$

μ is the marginal cost of utility $\mu = \Delta(p_1 q_1 + p_2 q_2) / \Delta U^*$.

- ▶ The minimum cost specification in Equation 1 follows the same concept

$$L = w_1 x_1 + w_2 x_2 + \lambda (Y^* - f(x_1, x_2)) \quad (8)$$

here $\lambda = \Delta(w_1 x_1 + w_2 x_2) / \Delta Y^*$ is the marginal cost of producing an additional unit of Y at Y^* .

- Several of the graphical interpretations follow. Consider the quadratic production function

$$f(x_1, x_2) = 5.0x_1 + 6.5x_2 + \frac{1}{2}(-0.50x_1^2 + 0.15x_1x_2 - 0.75x_2^2) \quad (9)$$

presented in Figure 1.

Production Function with Two Inputs

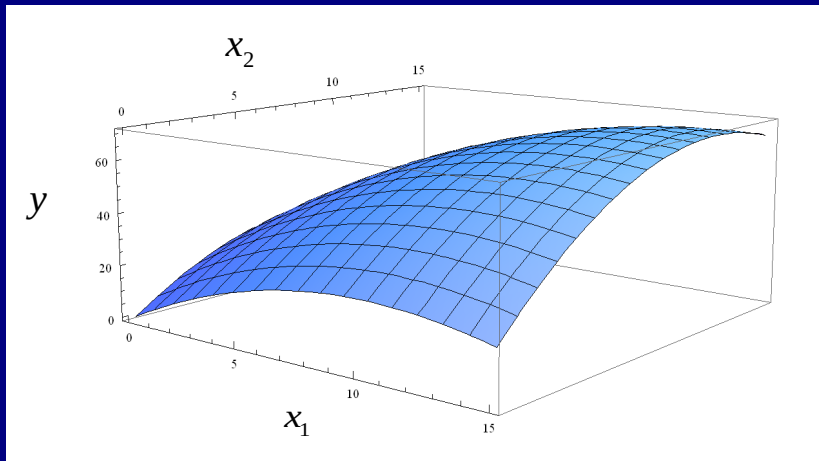


Figure : Production Function with Two Inputs

- ▶ The choice of inputs then follows the same process as the cost minimization for utility

$$\begin{aligned}\frac{\Delta L}{\Delta x_1} &= w_1 - \lambda \frac{\Delta f(x_1, x_2)}{\Delta x_1} = 0 \\ \frac{\Delta L}{\Delta x_2} &= w_2 - \lambda \frac{\Delta f(x_1, x_2)}{\Delta x_2} = 0\end{aligned}\quad (10)$$

- ▶ Solving each of the expressions in Equation 10 for the input price and taking the ratio of the results implies

$$\frac{\frac{\Delta f(x_1, x_2)}{\Delta x_1}}{\frac{\Delta f(x_1, x_2)}{\Delta x_2}} = \frac{\Delta x_2}{\Delta x_1} \Big|_{y=y^*} = \frac{w_1}{w_2} \quad (11)$$

where $\Delta x_2 / \Delta x_1|_{y=y^*}$ is referred to as the marginal rate of technical substitution between inputs.

Isoquant

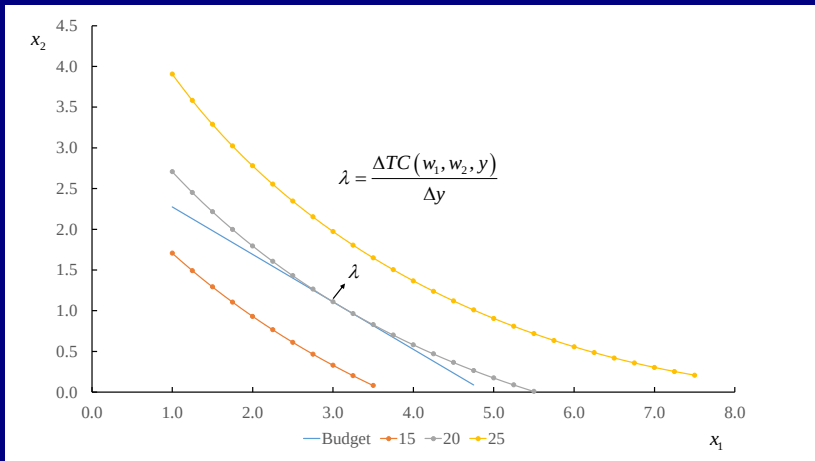


Figure : Isoquant Equilibrium

- The optimality conditions require

$$\begin{aligned}\frac{\Delta L}{\Delta x_1} &\Rightarrow x_1 = \frac{a_2 A_{12} \lambda - a_1 A_{22} \lambda + A_{22} w_1 - A_{12} w_2}{(A_{12}^2 - A_{11} A_{12}) \lambda} \\ \frac{\Delta L}{\Delta x_2} &\Rightarrow x_2 = \frac{a_1 A_{12} \lambda - a_2 A_{11} \lambda - A_{12} w_1 + A_{11} w_2}{(A_{12}^2 - A_{11} A_{12}) \lambda}\end{aligned}\quad (12)$$

$$\frac{\Delta L}{\Delta \lambda} \Rightarrow Y = a_1 x_1 + a_2 x_2 - \frac{1}{2} (A_{11} x_1^2 + A_{12} x_1 x_2 + A_{22} x_2^2)$$

- Substituting the first two results into the third result and then solving for λ yields

$$\lambda = \frac{\sqrt{A_{22} w_1^2 - 2 A_{12} w_1 w_2 + A_{11} w_2^2}}{\sqrt{A_{11} a_2^2 + A_{22} a_1^2 - 2 a_1 a_2 A_{12} - 2 A_{12}^2 y + 2 A_{11} A_{22} y}} \quad (13)$$

Don't worry, I don't expect you to be able to solve for these expression (in this class).

- ▶ However, notice that I could rewrite Equation 13 as

$$\lambda(w_1, w_2, y) = \frac{\sqrt{A_{22}w_1^2 - 2A_{12}w_1w_2 + A_{11}w_2^2}}{\sqrt{A_{11}a_2^2 + A_{22}a_1^2 - 2a_1a_2A_{12} - 2A_{12}^2y + 2A_{11}A_{22}y}} \quad (14)$$

Given that the rest of the parameters are constants of the production function, once I know w_1 , w_2 , and y I know the value of λ .

- ▶ If I substitute this value of $\lambda \Leftarrow \lambda(w_1, w_2, y)$ into each of the first two solutions in Equation 12, I will get an expression of x_1 and x_2 in terms of w_1 and w_2 .
- ▶ We denote these values as $x_1(w_1, w_2, y)$ and $x_2(w_1, w_2, y)$.

- ▶ Finally, I could substitute $x_1(w_1, w_2, y)$ and $x_2(w_1, w_2, y)$ into the cost portion (or the objective value) of Equation 3 to obtain

$$C(w_1, w_2, y) = w_1 x_1(w_1, w_2, y) + w_2 x_2(w_1, w_2, y) \quad (15)$$

- ▶ This cost function can be used to define the optimal level of production and input use

$$\begin{aligned} \max_y p_y y - C(w_1, w_2, y) &\Rightarrow p_y - \frac{\Delta C(w_1, w_2, y)}{\Delta y} \\ &\Rightarrow p_y - w_1 \frac{\Delta x_1(w_1, w_2, y)}{\Delta y} - w_2 \frac{\Delta x_2(w_1, w_2, y)}{\Delta y} \end{aligned} \quad (16)$$

- ▶ The second problem is similar, but does not require the Lagrangian

$$\max_{x_1, x_2} \pi = p_y \left(a_1 x_1 + a_2 x_2 + \frac{1}{2} (A_{11} x_1^2 + A_{12} x_1 x_2 + A_{22} x_2^2) \right) - w_1 x_1 - w_2 x_2. \quad (17)$$

- ▶ Taking the change in profit with respect to each variable

$$\begin{aligned} \frac{\Delta \pi}{\Delta x_1} &\Rightarrow x_1(p_y, w_1, w_2) = - \frac{A_{12} a_2 p_y - a_1 A_{22} p_y + A_{22} w_1 - A_{12} w_2}{(A_{12}^2 - A_{11} A_{22}) p_y} \\ \frac{\Delta \pi}{\Delta x_2} &\Rightarrow x_2(p_y, w_1, w_2) = - \frac{A_{12} a_1 p_y - a_2 A_{11} p_y - A_{12} w_1 + A_{11} w_2}{(A_{12}^2 - A_{11} A_{22}) p_y} \end{aligned} \quad (18)$$