

THE DIFFERENTIAL APPROACH TO CONSUMPTION THEORY

B1 Outline of the Approach

Let $p_1, \dots, p_n$ be the prices of n consumption goods (p as a column vector) and E the given total amount (or income) available to the consumer. His problem is to select the quantities $q_1, \dots, q_n$ (q as a column vector) that maximize a utility function $u(q)$ subject to a budget constraint $p'q = E$. The result is a system of n demand equations, each of which describes some q_i as a function of $n+1$ independent variables: $E, p_1, \dots, p_n$.

One way of deriving such a system is by specifying the algebraic form of the utility function (or of another function that emerges in the subsequent theory such as the consumer's indirect utility function or his cost function). The differential approach is different. It requires no initial algebraic specification of any function. Thus, the demand equations of this approach are completely general, *but they have coefficients that are not necessarily constant*. Obviously, the estimation of the equations requires some parameterization, but this comes only as the *last* step. This has the advantage of great flexibility.

The differential approach is based on the idea of taking the total differential of demand with respect to income and price variations. The approach involves the use of the Divisia (1925) price and volume indexes. These indexes are obtained by taking the total differential of the budget constraint, $E = p_1q_1 + \dots + p_nq_n$, with respect to all $2n$ prices and quantities.

$$dE = \sum_{i=1}^n q_i dp_i + \sum_{i=1}^n p_i dq_i.$$

We divide both sides by E and use the budget shares $w_i = p_i q_i / E$:

$$d(\log E) = \sum_{i=1}^n w_i d(\log p_i) + \sum_{i=1}^n w_i d(\log q_i)$$

we write this as

$$(1.1) d(\log E) = d(\log P) + d(\log Q)$$

where $d(\log P)$ is the Divisia price index and $d(\log Q)$ is the Divisia volume index, both differential form:

$$(1.2) d(\log P) = \sum_{i=1}^n w_i d(\log p_i)$$

$$(1.3) d(\log Q) = \sum_{i=1}^n w_i d(\log q_i).$$

Note that w_i in (1.2) and (1.3) is not a constant; this budget share is a function of the *levels* of income E and the price vector p . The other expressions that appear after the summation signs in (1.2) and (1.3), $d(\log p_i)$ and $d(\log q_i)$, are not levels but variations. We could indicate the dependence of the budget shares on E and p by writing then as $w_i(E, p)$, but we shall not do so because it would yield an unwieldy notation.

B2 Barten's Fundamental Matrix Equation

We write du/dq for the vector of first-order derivatives of the utility function (the marginal utilities of the n goods). We assume that these derivatives are all positive (nonsatiation). We write $U = d^2u/dq dq'$ for the Hessian matrix of the utility function; this U is assumed to be symmetric negative definite (generalized diminishing marginal utility).

To maximize $u(q)$ subject to the budget constraint $p'q = E$, we form the Lagrangian expression

$$(2.1) u(q) - \mu(p'q - E)$$

where μ is the Lagrangian multiplier. We differentiate (2.1) with respect to q :

$$(2.2) \frac{du}{dp} = \mu p.$$

This is the familiar proportionality of the marginal utilities and prices. This proportionality and the budget constraint constitute the first-order condition of the constrained utility maximum. The second-order condition is satisfied by the assumption of a negative definite Hessian U .

The i th equation of the proportionality (2.2) states that the marginal utility of good i equals μp_i . On dividing both sides of this equation by p_i we find that the marginal utility of a dollar spent on i equals μ , so that a one-dollar increase in income causes utility to rise by μ when this increase is spent on any of the n goods. Accordingly, μ is known as the *marginal utility of income*.

When we combine (2.2) with the budget constraint $p'q = E$, we have $n+1$ equations in $n+1$ unknowns: μ and the quantity vector q . The next step is to differentiate these $n+1$ equations with respect to the $n+1$ independent variables: income E and the price vector p . It was shown by Barten (1964) that the result can be written in the following partitioned matrix form:

$$\begin{bmatrix} U & p \\ p' & 0 \end{bmatrix} \begin{bmatrix} \frac{dq}{dE} & \frac{dq}{dp'} \\ -\frac{d\mu}{dE} & -\frac{d\mu}{dp'} \end{bmatrix} = \begin{bmatrix} 0 & \mu I \\ 1 & -q' \end{bmatrix}$$

The matrix on the far left, which is square of order $n+1$, is the bordered Hessian matrix of the utility function (bordered by prices). The matrix immediately to the left of the equal sign is also square of order $n+1$; it contains the derivatives of all dependent variables (q and μ) with respect to all independent variables (p and E). A step-by-step derivation of (2.3) is given in section B8 at the end of this appendix.

Next we solve the matrix equation (2.3) for the derivatives dq/dE and dq/dp' , for which purpose we premultiply both sides of the equation by the inverse of the bordered Hessian on the far left. Details are provided in Section B8. The main result is that dq/dp' can be written as the sum of three terms,

$$(2.4) \frac{dq}{dp'} = \mu U^{-1} - \phi E \frac{dq}{dE} \frac{dq'}{dE} - \frac{dq}{dE} q'$$

where ϕ is the *income flexibility*, i.e., the reciprocal of the income elasticity of the marginal utility of income:

$$(2.5) \frac{1}{\phi} = \frac{d\mu}{dE} \frac{E}{\mu}.$$

The three terms on the right in (2.4) are quite important for the developments which follow. The last, $-(dq/dE)q'$, is the *income effect* of the price change on demand. The two other terms represent the total substitution effect: the response

q_i to a change in p_j , with *real* income (and other prices) held constant. This total substitution effect consists of a *specific substitution effect*, μU^{-1} , and a *general substitution effect*,

$$-\phi E \frac{dq}{dE} \frac{dq'}{dE}$$

with the terminology from Houthakker (1960). The general effect is concerned with the competition of all n goods for an extra dollar of consumer's income, whereas the specific effect deals with the interaction of the two goods (i and j) in the utility function. The specific effect takes the form $\mu u^{i,j}$ for these two goods, $u^{i,j}$ being the (i,j) th element of the matrix U^{-1} .

B3 A General Form of the Differential Demand System

When we have solved (2.3) for the derivatives dq/dE and dq/dp' , we can combine them to obtain the total differential of demand, dq . Thus, we can express dq as the sum of dE multiplied by dq/dE and dp premultiplied by dq/dp' . It will appear, however, that a much simpler and clearer algebraic picture can be obtained when we choose a formulation closer to the Divisia decomposition.

Consider the total differential of the budget share $w_i - p_i q_i / E$:

$$dw_i = \frac{q_i}{E} dp_i + \frac{p_i}{E} dq_i - \frac{p_i q_i}{E^2} dE.$$

We write this in the more convenient form

$$(3.1) dw_i = w_i d(\log p_i) + w_i d(\log q_i) - w_i d(\log E)$$

which shows that the change in the budget share is the sum of three components: a price component, a quantity component, and an income (or total expenditure) component.

Note that the quantity component $w_i d(\log q_i)$ is equal to the contribution of good i to the Divisia volume index shown in (1.3). This expression is also the dependent variable in the following differential demand system:

$$(3.2) w_i d(\log q_i) = \theta_i d(\log Q) + \phi \sum_{j=1}^n \theta_{i,j} d \left[\log \frac{p_j}{p^*} \right]$$

The remainder of this section provides an explanation of the terms on the right. Again, the mathematical derivations are postponed until Section B8.

The first term on the right, $\theta_i d(\log Q)$, is the real-income term of demand, with θ_i defined as the marginal share of good i . Real income is thus measured by the Divisia volume index. The term results from two sources: the change in money income E and the income effect of the price changes. More precisely, the income effect of the price changes on the dependent variable in (3.2) is equal to θ_i multiplied by the Divisia price index $d(\log P)$; this effect transforms the change in money income into a real-income change.

The other term on the right in (3.2) is the substitution term. The price of good j occurs in deflated for,

$$(3.3) d \left[\log \frac{p_j}{p^*} \right] = d(\log p_j) - d(\log P^*)$$

and the deflator is the Frisch (1932) price index,

$$(3.4) d(\log P^*) = \sum_{i=1}^n \theta_i d(\log p_i)$$

which uses marginal shares as weights. Since $\theta_i > w_i$ holds for luxuries and $\theta_i < w_i$ for necessities, the Frisch price index gives a larger (smaller) weight to the logarithmic price changes of luxuries (necessities) than does the Divisia price index.

The substitution term in (3.2) represents the total substitution effect of the price changes. The specific component takes the form

$$(3.5) \phi \sum_{j=1}^n \theta_{ij} d(\log p_j)$$

which is the undeflated form of the substitution term in (3.2). It is the general substitution effect of the price changes that is represented by the Frisch price index. Thus, the income effect of the price changes deflates the change in money income, while the general substitution effect of the price changes deflates their specific substitution effect; the form deflator is the Divisia price index and the latter is the Frisch price index.

The θ'_{ij} s in the substitution term of (3.2) take the form

$$(3.6) \theta_{ij} = \frac{\mu}{\phi E} p_i u^{ij} p_j$$

where u^{ij} is the (i,j)th element of U^{-1} . The $n \times n$ matrix $[\theta_{ij}]$ is symmetric positive definite. Summation of θ_{ij} over j gives the i th marginal share:

$$(3.7) \sum_{j=1}^n \theta_{ij} = \theta_i.$$

Since the θ'_i s sum to 1 (sum over $i=1, \dots, n$), so do the θ'_{ij} s when summed over all i and j .

B4 Slutsky Coefficients

In (3.2) we use the general substitution effect of the price changes to deflate the specific substitution effect of these changes, but it is also possible to present the total substitution effect directly without separating its two components. This is a matter of combining p_j and its deflator. The reader should verify, using (3.3), (3.4), (3.7) as well as symmetry of the matrix $[\theta_{ij}]$, that (3.2) can be simplified to

$$(4.1) w_i d(\log q_i) = \theta_i d(\log Q) + \sum_{j=1}^n \pi_{ij} d(\log p_j)$$

where

$$(4.2) \pi_{ij} = \phi(\theta_{ij} - \theta_i \theta_j)$$

which is known as the (i,j)th Slutsky coefficient. The $n \times n$ matrix $[\pi_{ij}]$ is symmetric

$$(4.3) \pi_{ij} = \pi_{ji}$$

and it is negative semidefinite of rank $n-1$. The cause of the singularity is

$$(4.4) \sum_{j=1}^n \pi_{ij} = 0, i = 1, \dots, n$$

which follows directly from (3.7). Condition (4.3) is known as *Slutsky symmetry*, and (4.4) as *demand homogeneity*. The latter property states that demand remains unchanges when all prices and money income change proportionately (so that real income does not change).

Recall from Section B1 (second paragraph) that the coefficients of the differential approach need not be constant. This applies here everywhere. In particular, the θ'_i s, and π'_{ij} s in (4.1) and the θ'_i s, θ'_{ij} s and ϕ in (3.2) should be viewed as functions of the levels of income and prices. The parameterization is aseperate decision. One such decision will be discussed now; other decisions will be considered in Sectons B5 and B7.

Imagine that we have data for a number of successive time periods: $t=1,2,\dots$. We define

$$(4.5) Dx_t = \log x_t - \log x_{t-1} = \log \frac{x_t}{x_{t-1}}$$

as the finite log-chage from period $t-1$ to period t of any positive variable x . We approximate $d(\log q_i)$ on the left in (4.1) by Dq_{it} and $d(\log p_j)$ on the right by Dp_{jt} . For w_i on the left we could use either w_{it} or w_{it-1} ; the simplest choice is to take the average, $\bar{w}_{it} = \frac{1}{2}(w_{i,t-1} + w_{i,t})$. We thus obtain a finite-change version of (4.1) of the form

$$(4.6) \bar{w}_{it} Dq_{it} = \theta_i DQ_t + \sum_{j=1}^n \pi_{ij} Dp_{jt} + \epsilon_{it}$$

where ϵ_{it} is a random error term and $DQ_t = \sum_{i=1}^n \bar{w}_{it} Dq_{it}$ is a finite-change version of the Divisia volume index.

Equation (4.6) with constant θ'_i s and π'_{ij} s is the i th equation of the *Rotterdam model in absolute prices*. (A relative price version will be considered in Section B5.) This model originated with Barten (1964) and Theil (1965); we refer to Barnett (1979,1984), Byron (1984) and Mountain (1988) for further analysis of the approximations involved. The simulation experiments of Laitinen (1978) and Meisner (1979) described in Section 3.3 are based on the parameterization (4.6) with constant θ'_i s and π'_{ij} s.

B5 Preference Independence

Preference independence means that the consumer's tastes can be represented by a utility function equal to the sum of n functions, one for each quantity:

$$(5.1) u(q) = u_1(q_1) + \dots u_n(q_n).$$

The marginal utility of each good i is now a function of q_i only, so that both the Hessian U and its inverse are diagonal matrices. It thus follows from (3.6) that $[\theta_{ij}]$ is also diagonal and from (3.7) that the diagonal element θ_{ij} is equal to the marginal share θ_i . Therefore, under preference independence we can simplify (3.2) to

$$(5.2) w_i d(\log q_i) = \theta_i d(\log Q) + \phi \theta_i d \left[\log \frac{p_i}{P^*} \right]$$

which contains only the Frisch-deflated price of good i . It is readily verified that (5.2) is a special case of (4.1) with the Slutsky coefficients specified as

$$\begin{aligned} (5.3) \pi_{ij} &= \phi \theta_i (1 - \theta_i) \text{ if } i = j \\ &= -\phi \theta_i \theta_j \text{ if } i \neq j. \end{aligned}$$

As before, the implementation of a differential demand system requires a decision on the parameterization. Two such decisions will now be discussed. Going back to the notation developed in the discussion of equation (4.5) and (4.6), we approximate (5.2) by

$$(5.4) \bar{w}_{it} Dq_{it} = \theta_i DQ_t + \phi \theta_i (Dp_{it} - DP_t^*) + \epsilon_{it}$$

where $DP_t^* = \sum_{i=1}^n \theta_i Dp_{it}$ is the finite-change version of the Frisch price index. This model with constant θ_i 's and ϕ is the *relative price version of the Rotterdam model under preference independence*. Obviously, (5.4) is a special case of (4.6) with the Slutsky coefficients specified as shown in (5.3).

Alternatively, let the θ_i 's in (5.4) be constant but let ϕ have a subscript t ,

$$(5.5) \bar{w}_{it} Dq_{it} = \theta_i DQ_t + \phi_t \theta_i (Dp_{it} - DP_t^*) + \epsilon_{it}$$

where

$$(5.6) \phi_t = \phi_{t-1} + c DQ_t.$$

In this model the income flexibility is not a constant. Instead, the change in this flexibility is proportional to the change in real income (as measured by the finite-change version of the Divisia volume index). Frisch (1959) conjectured that ϕ takes larger absolute values for the rich than for the poor. In the present parameterization, consisting of (5.5) and (5.6) with constant c and θ_i 's, this would amount to $c \neq 0$. Note that (4.6) does not contain this model as a special when $c \neq 0$.

The additivity assumption (5.1) can be made less restrictive by applying it to groups of goods rather than single goods. The developments based on this weaker assumption include composite differential demand equations for groups of goods as well as conditional differential demand equations for goods within their group. Details on these developments are provided in Chapter 6 (starting in Section (6.4)).

B6 The Price Terms of the Working-PI Model

We now return to equation (8.2) of Section 2.8, reproduced here:

$$(6.1) w_{ic} = \alpha_i + \beta_i \log Q_c + (w_{ic} - \tilde{w}_{ic}) + \epsilon_{ic}.$$

Recall that the difference $w_{ic} - \tilde{w}_{ic}$ on the right results from the move from the price vector $[\bar{p}_i]$ to $[p_{ic}]$, real income being fixed at the level Q_c . We proceed to apply the differential approach in order to find the effect of these price changes on the budget share.

For this purpose we write equation (3.1) of Section B3 as

$$(6.2) dw_i = w_i [d(\log p_i) - d(\log P)] + w_i d(\log q_i) \\ - w_i [d(\log E) - d(\log P)]$$

where $d(\log P)$ is the Divisia price index defined in equation (1.2) of section B1. It follows from (1.1) that the expression in brackets in the second line of (6.2) is the Divisia volume index $d(\log Q)$. In the present context this $d(\log Q)$ vanishes because real income is fixed (at the level Q_c). Therefore, (6.2) is simplified to

$$(6.3) dw_i = w_i [d(\log p_i) - d(\log P)] + w_i d(\log q_i)$$

The expression on the far right is the dependent variable of the differential demand equation (4.1). The real-income term in (4.1) vanishes so that (6.3) can now be expressed in terms of price changes only:

$$(6.4)dw_i = w_i[d(\log p_i) - d(\log P)] + \sum_{j=1}^n \pi_{ij}d(\log p_j).$$

The substitution term on the far right can then be simplified to that in (5.2) under preference independence. The result given in equation (8.3) of Section 2.8 is then immediately obtained by interpreting $d(\log p_i)$ as $\log p_{it} - \log \bar{p}_i$ and applying the mean value theorem of differential calculus.

B7 A Time Series Version of the Working-PI Model

Recall from Section (2.2) that, under Working's model, the marginal share of good i exceeds the corresponding budget share by the constant β_i . In a time series context, we select $\bar{w}_{it}$ (the average of $w_{i,t-1}$ and w_{it}) for the budget share during the transition period $t-1$ and t . Thus, the marginal share of good i takes the form $\bar{w}_{it} + \beta_i$. Substitution of this expression for θ_i in (5.4) yields, after minor rearrangements,

$$(7.1)\bar{w}_{it}(Dq_{it} - DQ_t) = \beta_i DQ_t + \phi(\bar{w}_{it} + \beta_i)(Dp_{it} - DP_t^*) + \epsilon_{it}$$

where the Frisch price index now takes the following form:

$$(7.2)DP_t^* = \sum_{i=1}^n (\bar{w}_{it} + \beta_i)Dp_{it}.$$

Equation (7.1) with constant ϕ and β_i 's may be viewed as a time series version of the Working-PI model. Both here and in (5.4), the Frisch price index is linear in the parameters: linear in the β_i 's here, and in the θ_i 's in (5.4). Also, both in (5.4) and in (7.1), the real-income term is linear but the substitution term is cubic in the parameters. The Japanese elasticities according to Working's model in Table 5-1 (Section 5.1) are based on the specification (7.1).

Additional insight is obtained when we combine (3.1) and (1.1) in the form

$$(7.3)dw_i = w_i[d(\log p_i) - d(\log P)] + w_i[d(\log q_i) - d(\log Q)].$$

We write the finite-change version of (7.3) as

$$(7.4)w_{it} - w_{i,t-1} = \bar{w}_{it}(Dp_{it} - DP_t) + \bar{w}_{it}(Dq_{it} - DQ_t)$$

where $DP_t = \sum_{i=1}^n \bar{w}_{it}Dp_{it}$ is a finite-change version of the Divisia price index. Then, by adding $\bar{w}_{it}(Dp_{it} - DP_t)$ to both sides of (7.1), we obtain

$$(7.5)w_{it} - w_{i,t-1} = \beta_i DQ_t + \bar{w}_{it}(Dp_{it} - DP_t) + \phi(\bar{w}_{it} + \beta_i)(Dp_{it} - DP_t^*) + \epsilon_{it}$$

which should be compared with the cross-country model given in equations (3.4) to (3.7) of Section (2.3). Both are demand systems based on Working's model with a substitution term under preference independence (with a constant income flexibility). Both systems have a real-income term, a pure price term and a substitution term in the right side of each equation. The difference is that the dependent variable in (7.5) is the change in (not the level of) a budget share and that the pure price

term in (7.5) does not involve unknown parameters, whereas the corresponding term in the cross-country model is quadratic in the parameters.

We shall return to (7.1) and (7.5) in Appendix C.

B8 Derivations

The Fundamental Matrix Equation

We differentiate the budget constraint $p'q = E$ with respect to p_j and E :

$$\sum_{i=1}^n p_i \frac{\partial q_i}{\partial p_j} = -q_j \text{ and } \sum_{i=1}^n \frac{\partial q_i}{\partial E} = 1.$$

We can write this result in matrix form ($j=1, \dots, n$) as

$$(8.1) p'(dq/dp') = -q', p'(dq/dE) = 1$$

where $dq/dp' = [\partial q_i / \partial p_j]$ is the $n \times n$ matrix of price derivatives of demand and $dq/dE = [\partial q_i / \partial E]$ is the column vector of n income derivatives.

Next we differentiate the proportionality (2.2) with respect to p_j and E to give

$$\sum_{k=1}^n \frac{\partial^2 u}{\partial q_i \partial q_k} \frac{\partial q_k}{\partial p_j} = \mu \delta_{ij} + p_i \frac{\partial \mu}{\partial p_j}$$

where δ_{ij} is the Kronecker delta (=1 if $i=j$, 0 otherwise), and

$$\sum_{k=1}^n \frac{\partial^2 u}{\partial q_i \partial q_k} \frac{\partial q_k}{\partial E} = p_i \frac{d\mu}{dE}.$$

We can write these results in matrix form as

$$(8.2) U(dq/dp') = \mu I + p(d\mu/dp'), U(dq/dE) = (d\mu/dE)p$$

where $d\mu/dp' = [\partial \mu / \partial p_j]$. The matrix equation (2.3) is then obtained by combining the four equations in (8.1) and (8.2).

Solving the Matrix Equation

The inverse of the bordered Hessian on the far left in (2.3) is

$$\begin{bmatrix} U & p \\ p' & 0 \end{bmatrix}^{-1} = \frac{1}{p'U^{-1}p} \begin{bmatrix} (p'U^{-1}p)U^{-1} - U^{-1}p(U^{-1}p)' & U^{-1}p \\ (U^{-1}p)' & -1 \end{bmatrix}$$

Using this inverse we obtain the following solution:

$$\begin{bmatrix} dq/dE & dq/dp' \\ -d\mu/dE & -d\mu/dp' \end{bmatrix} = \frac{1}{p'U^{-1}p} \begin{bmatrix} (p'U^{-1}p)U^{-1} - U^{-1}p(U^{-1}p)' & U^{-1}p \\ (U^{-1}p)' & -1 \end{bmatrix} \begin{bmatrix} 0 & \mu I \\ 1 & -q' \end{bmatrix}.$$

Carrying out the matrix multiplication block by block, we obtain

$$(8.3) \frac{dq}{dE} = \frac{1}{p'U^{-1}p} U^{-1}p$$

$$(8.4) \frac{dq}{dp'} = \mu U^{-1} - \frac{\mu}{p'U^{-1}p} U^{-1} p ((U^{-1}p)') - \frac{1}{p'U^{-1}p} U^{-1} p q'$$

$$(8.5) \frac{d\mu}{dE} = \frac{1}{p'U^{-1}p}$$

$$(8.6) \frac{d\mu}{dp} = -\frac{\mu}{p'U^{-1}p} U^{-1} p - \frac{1}{p'U^{-1}p} q$$

To simplify these expressions we use (8.5) to substitute $d\mu/dE$ for the reciprocal of $p'U^{-1}p$ in (8.3) and (8.4). Equation (8.3) then becomes

$$(8.7) \frac{dq}{dE} = \frac{d\mu}{dE} U^{-1} p.$$

It is then readily verified from (2.5) and (8.7) that (8.4) can be written in the form (2.4).

The Differential Demand System

To verify the differential demand system (3.2) we take

$$dq_i = \frac{\partial q_i}{\partial E} dE + \sum_{j=1}^n \frac{\partial q_i}{\partial p_j} dp_j$$

as our starting point. We transform this equation to logarithmic differential form by multiplying both sides by p_i/E and using $w_i = p_i q_i/E$:

$$w_i d(\log q_i) = \frac{\partial(p_i q_i)}{\partial E} d(\log E) + \sum_{j=1}^n \frac{p_i p_j}{E} \frac{\partial q_i}{\partial p_j} d(\log p_j).$$

Since the derivative immediately to the right of the equal sign is the marginal share θ_i , we can simplify this result to

$$(8.8) w_i d(\log q_i) = \theta_i d(\log E) + \sum_{j=1}^n \frac{p_i p_j}{E} \frac{\partial q_i}{\partial p_j} d(\log p_j).$$

The derivative $\partial q_i/\partial p_j$ in the price term of (8.8) is the (i,j)th element of the matrix (2.4). Therefore, we can write this term as

$$\sum_{j=1}^n \frac{p_i p_j}{E} \left[\mu u^{ij} - \phi E \frac{\partial q_i}{\partial E} \frac{\partial q_j}{\partial E} - \frac{\partial q_i}{\partial E} q_j \right] d(\log p_j) = \sum_{j=1}^n (\phi \theta_{ij} - \phi \theta_i \theta_j - \theta_i w_j) d(\log p_j)$$

where use is made of the definition (3.6) of θ_{ij} . Substitution of this result into (8.8) yields

$$w_i d(\log q_i) = \theta_i \left[d(\log E) - \sum_{j=1}^n w_j d(\log p_j) \right] + \phi \sum_{j=1}^n \theta_{ij} d(\log p_j) - \phi \theta_i \sum_{j=1}^n \theta_j d(\log p_j).$$

It follows from (1.1) and (1.2) that the expression in brackets in the first term on the right equals $d(\log Q)$. The result (3.2) is then confirmed when we succeed in verifying (3.7). This verification is a matter of combining (3.6) and (8.7).

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