

INVERTING BARTEN'S FUNDAMENTAL MATRIX AND CONSTRUCTION OF THE ROTTERDAM MODEL

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ABSTRACT. In the spring of 1996 Henri Theil collaborated with Charles B. Moss and James L. Seale to over a seminar class at the University of Florida involving the derivation and application of differential demand systems including the Rotterdam and Florida Models. This paper demonstrates how the basic differential model can be derived from the inverse of Barten's fundamental equation.

1. INTRODUCTION

My previous lecture on the derivation of the differential demand system set up the basic concept of differentiating the first-order conditions for the utility model. In this lecture, we start with the basic differential system in Equation 2.13.

2. BASIC FORMULATION OF BARTEN'S FUNDAMENTAL MATRIX

In Lecture 1 we differentiated the first-order conditions of the consumer's utility function subject to the budget constraint to yields Barten's fundamental matrix equation

$$(2.1) \quad \begin{bmatrix} U & p \\ p' & 0 \end{bmatrix} \begin{bmatrix} \frac{dq}{dE} & \frac{dq}{dp'} \\ -\frac{d\mu}{dE} & -\frac{d\mu}{dp'} \end{bmatrix} = \begin{bmatrix} 0 & \mu I \\ 1 & -q' \end{bmatrix}$$

(e.g., Equation 2.13 of Lecture 1 - Notes on the Differential Approach – Deriving the Change in First-Order Conditions). The next challenge will be to use these results to construct the differential demand relationships. Noting that Barten's fundamental matrix equation can be

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solved for dq_i/dE and dq_i/dp_j we differentiate the quantity q_i consumed with respect to each exogenous variable yielding

$$(2.2) \quad dq_i = \frac{\partial q_i}{\partial E} dE + \sum_{j=1}^n \frac{\partial q_i}{\partial p_j} dp_j.$$

To transform this expression into the traditional differential demand form we note that

$$(2.3) \quad d(\log x) = \frac{1}{x} dx$$

therefore

$$(2.4) \quad x d(\log x) = dx.$$

Substituting this result for dq_i , dE , and dp_j we have

$$(2.5) \quad q_i d(\log q_i) = \frac{\partial q_i}{\partial E} E d(\log E) + \sum_{j=1}^N p_j \frac{\partial q_i}{\partial p_j} d(\log p_j).$$

Note that $\partial q_i/\partial E$ and $\partial q_i/\partial p_j$ terms are from Barten's fundamental matrix. Multiplying each side of this expression by p_i/E yields

$$(2.6) \quad w_i d(\log q_i) = \frac{\partial (p_i q_i)}{\partial E} d(\log E) + \sum_{j=1}^n \frac{p_i p_j}{E} \frac{\partial q_i}{\partial p_j} d(\log p_j).$$

Defining $\partial (p_i q_i)/\partial E$ as the marginal share of income θ_i we are left with

$$(2.7) \quad w_i d(\log q_i) = \theta_i d(\log E) + \sum_{j=1}^n \frac{p_i p_j}{E} \frac{\partial q_i}{\partial p_j} d(\log p_j).$$

Thus, to finish out the derivation, we must define dq_i/dp_j by inverting Barten's fundamental matrix

$$\begin{aligned}
 (2.8) \quad & \left[\begin{array}{cc|cc} U & p & I & 0 \\ p' & 0 & 0 & 1 \end{array} \right] U^{-1}R_1 \\
 & \left[\begin{array}{cc|cc} I & U^{-1}p & U^{-1} & 0 \\ p' & 0 & 0 & 1 \end{array} \right] R_2 - p'R_1 \\
 & \left[\begin{array}{cc|cc} I & U^{-1}p & U^{-1} & 0 \\ 0 & -p'U^{-1}p & -p'U^{-1} & 1 \end{array} \right] -\frac{1}{p'U^{-1}p}R_2 \\
 & \left[\begin{array}{cc|cc} I & U^{-1} & U^{-1} & 0 \\ 0 & 1 & \frac{1}{p'U^{-1}p}p'U^{-1} & -\frac{1}{p'U^{-1}p} \end{array} \right] R_1 - U^{-1}pR_2 \\
 & \left[\begin{array}{cc|cc} I & 0 & U^{-1} - \frac{1}{p'U^{-1}p}(U^{-1}p)(p'U^{-1}) & \frac{1}{p'U^{-1}p}U^{-1}p \\ 0 & 1 & \frac{1}{p'U^{-1}p}p'U^{-1} & -\frac{1}{p'U^{-1}p} \end{array} \right]
 \end{aligned}$$

Therefore, the solution to the matrix differentiation is

$$\begin{aligned}
 (2.9) \quad & \left[\begin{array}{cc} \frac{dq}{dE} & \frac{dq}{dp'} \\ -\frac{d\mu}{dE} & \frac{d\mu}{dp'} \end{array} \right] = \\
 & \frac{1}{p'U^{-1}p} \left[\begin{array}{cc} (p'U^{-1}p)U^{-1} - (U^{-1}p)U^{-1} - (U^{-1}p)(p'U^{-1}) & U^{-1}p \\ p'U^{-1} & -1 \end{array} \right] \cdot \\
 & \times \left[\begin{array}{cc} 0 & \mu I \\ 1 & -q' \end{array} \right]
 \end{aligned}$$

Solving for $d\mu/dE$ first, we get

$$(2.10) \quad \frac{d\mu}{dE} = \frac{1}{p'U^{-1}p}.$$

Next, we solve for dq/dE

$$(2.11) \quad \frac{dq}{dE} = \frac{1}{p'U^{-1}p}U^{-1}p$$

substituting in the $d\mu/dE = 1/(p'U^{-1}p)$ we have

$$(2.12) \quad \frac{dq}{dE} = \frac{d\mu}{dE}U^{-1}p.$$

Next we turn to dq/dp' which is defined by the matrix multiplication as

$$(2.13) \quad \frac{dq}{dp'} = \mu U^{-1} - \frac{\mu}{p' U^{-1} p} (U^{-1} p) (U^{-1} p)' - \frac{1}{p' U^{-1} p} U^{-1} p q'$$

First, focusing on the center term

$$(2.14) \quad \frac{\mu}{p' U^{-1} p} (U^{-1} p) (U^{-1} p)' = \mu (p' U^{-1} p) \left(\frac{1}{p' U^{-1} p} U^{-1} p \right) \left(\frac{1}{p' U^{-1} p} (U^{-1} p)' \right)$$

which yields

$$(2.15) \quad \frac{\mu}{p' U^{-1} p} (U^{-1} p) (U^{-1} p)' = \mu (p' U^{-1} p) \left(\frac{d1}{dE} \right) \left(\frac{dq}{dE} \right)'$$

(i.e., by Equation 2.12). Further, given that

$$(2.16) \quad \frac{1}{\phi} = \frac{d\mu}{dE} \frac{E}{\mu}$$

the center expression then becomes

$$(2.17) \quad \frac{\mu}{p' U^{-1} p} (U^{-1} p) (U^{-1} p)' = \phi E \left(\frac{dq}{dE} \right) \left(\frac{dq}{dE} \right)'.$$

The complete derivation then becomes

$$(2.18) \quad \frac{dq}{dp'} = \mu U^{-1} - \phi E \frac{dq}{dE} \frac{dq'}{dE} - \frac{dq}{dE} q'$$

To finish the derivation, we start from Equation 2.6

$$(2.19) \quad \begin{aligned} w_i d(\log q_i) &= \frac{\partial (p_i q_i)}{\partial E} d(\log E) + \sum_{j=1}^n \frac{p_i p_j}{E} \frac{\partial q_i}{\partial p_j} d(\log p_j) \\ \theta_i d(\log E) &+ \sum_{j=1}^n \left(\frac{p_i p_j}{E} \mu u^{ij} - \phi \frac{\partial (p_i q_i)}{\partial E} \frac{\partial (p_j q_j)}{\partial E} - \frac{\partial (p_i q_i)}{\partial E} \frac{p_j q_j}{E} \right) d(\log p_i). \end{aligned}$$

Substituting $\theta_{ij} = \mu p_i u^{ij} p_j / \phi E$ and for the marginal shares yields

$$(2.20) \quad w_i d(\log q_i) = \theta_i d(\log E) + \sum_{j=1}^n (\phi \theta_{ij} - \phi \theta_i \theta_j - \theta_i w_j) d(\log p_j).$$

Breaking the sum up we are left with

$$(2.21) \quad w_i d(\log q_i) = \theta_i \left(d(\log E) - \sum_{j=1}^n w_j d(\log p_j) \right) + \phi \sum_{j=1}^n (\theta_{ij} - \theta_i \theta_j) d(\log p_j).$$

Given that

$$(2.22) \quad d(\log E) - \sum_{j=1}^n w_j d(\log p_j) = d(\log Q)$$

we are left with the general differential model

$$(2.23) \quad w_i d(\log q_i) = \theta_i d(\log Q) + \phi \sum_{j=1}^n (\theta_{ij} - \theta_i \theta_j) d(\log p_j).$$

REFERENCES

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