

NOTES ON THE DIFFERENTIAL APPROACH – DERIVING THE CHANGE IN FIRST ORDER CONDITIONS

CHARLES B. MOSS

ABSTRACT. In the spring of 1996 Henri Theil collaborated with Charles B. Moss and James L. Seale to offer a seminar class at the University of Florida involving the derivation and application of differential demand systems including the Rotterdam and Florida Models. This paper presents the derivation of Barten's fundamental equation in matrix form which underlies both models.

1. INTRODUCTION

The literature contains several procedures for parameterizing demand systems for commodities from *ad hoc* to the restrictive. For example, one could start with a particular utility function form such as the Cobb-Douglas and then derive the implied demand equations. A slightly less restrictive approach is to start with a general expenditure function and derive the associated demand curves by the appropriate envelope theorems. One such formulation is the Almost Ideal Demand System (AIDS). However, each of these formulations impose certain restrictions on the demand functions.

2. DEFINING THE DIFFERENTIAL OF THE FIRST ORDER CONDITIONS

An alternative formulation is the differential approach to demand which essentially parameterizes the demand system by quantifying changes in optimizing behavior. The beginning point of the differential approach is the Lagrangian for utility maximization subject to the budget constraint

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$$(2.1) \quad u(q) + \mu(E - p'q)$$

where q is a vector of n quantities consumed, p is a conformable vector of prices, E is the total level of expenditures, and μ is the Lagrange multiplier which parameterizes the marginal utility of income. The first order conditions for this problem are then

$$(2.2) \quad \begin{bmatrix} \frac{\partial u}{\partial q_1} - \mu p_1 \\ \frac{\partial u}{\partial q_2} - \mu p_2 \\ \vdots \\ \frac{\partial u}{\partial q_n} - \mu p_n \\ E - p'q \end{bmatrix} = 0$$

Restricting our attention to the first element in the matrix, we could ask: How does this first order condition change as the k^{th} price changes? Differentiating the first element of the first order condition matrix with respect to p_k yields

$$(2.3) \quad \frac{\partial^2 u}{\partial q_1 \partial q_1} \frac{dq_1}{dp_k} + \frac{\partial^2 u}{\partial q_1 \partial q_2} \frac{dq_2}{dp_k} + \cdots \frac{\partial^2 u}{\partial q_1 \partial q_n} \frac{dq_n}{dp_k} - \delta_{1k} \mu - p_k \frac{\partial \mu}{\partial p_k} = 0$$

where δ_{ij} is the Kronecker delta which is equal to 1 if $i = j$ and 0 otherwise. Hence, if $k = 1$ this equation becomes

$$(2.4) \quad \frac{\partial^2 u}{\partial q_1 \partial q_1} \frac{q_1}{p_1} + \frac{\partial^2 u}{\partial q_1 \partial q_2} \frac{dq_2}{p_1} + \cdots \frac{\partial^2 u}{\partial q_1 \partial q_n} \frac{dq_n}{p_1} - \mu - p_1 \frac{\partial \mu}{\partial p_1} = 0$$

otherwise, the expression is

$$(2.5) \quad \frac{\partial^2 u}{\partial q_1 \partial q_1} \frac{q_1}{p_k} + \frac{\partial^2 u}{\partial q_1 \partial q_2} \frac{dq_2}{dp_k} + \cdots \frac{\partial^2 u}{\partial q_1 \partial q_n} \frac{dq_n}{dp_k} - p_k \frac{\partial \mu}{\partial p_k} = 0$$

The first part of this expression can be written in matrix form as

$$(2.6) \quad \begin{bmatrix} \frac{\partial^2 u}{\partial q_1 \partial q_1} & \frac{\partial^2 u}{\partial q_1 \partial q_2} & \cdots & \frac{\partial^2 u}{\partial q_1 \partial q_n} \end{bmatrix} \begin{bmatrix} \frac{dq_1}{dp_k} \\ \frac{dq_2}{dp_k} \\ \vdots \\ \frac{dq_n}{dp_k} \end{bmatrix}$$

Thus, in general matrix notation we let

$$(2.7) \quad U = \begin{bmatrix} \frac{\partial^2 u}{\partial q_1 \partial q_1} & \frac{\partial^2 u}{\partial q_1 \partial q_2} & \cdots & \frac{\partial^2 u}{\partial q_1 \partial q_n} \\ \frac{\partial^2 u}{\partial q_2 \partial q_1} & \frac{\partial^2 u}{\partial q_2 \partial q_2} & \cdots & \frac{\partial^2 u}{\partial q_2 \partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 u}{\partial q_n \partial q_1} & \frac{\partial^2 u}{\partial q_n \partial q_2} & \cdots & \frac{\partial^2 u}{\partial q_n \partial q_n} \end{bmatrix}$$

In addition we define

$$(2.8) \quad \frac{dq}{dp'} = \begin{bmatrix} \frac{dq_1}{dp_1} & \frac{dq_1}{dp_2} & \cdots & \frac{dq_1}{dp_n} \\ \frac{dq_2}{dp_1} & \frac{dq_2}{dp_2} & \cdots & \frac{dq_2}{dp_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{dq_n}{dp_1} & \frac{dq_n}{dp_2} & \cdots & \frac{dq_n}{dp_n} \end{bmatrix}$$

$U(dq/dp')$ is then a $n \times n$ matrix depicting the first part of the differential of the first order conditions with respect to p_k . Remembering the Kronecker deltas, the complete differential of the first order conditions with respect to p_k then becomes

$$(2.9) \quad U \frac{dq}{dp'} = \mu I + p \frac{d\mu}{dp'}.$$

Defining $d\mu/dp'$ similarly to dp/dp' as

$$(2.10) \quad \frac{d\mu}{dp'} = \begin{bmatrix} \frac{d\mu}{dp_1} & \frac{d\mu}{dp_2} & \cdots & \frac{d\mu}{dp_n} \end{bmatrix}$$

we know that $pd\mu/dp'$ is an $n \times n$ matrix conformable in addition.

Next, we turn to differentiating the first order conditions with respect to total income (E). Again taking the first element in the first order condition matrix we have

$$(2.11) \quad \frac{\partial^2 u}{\partial q_1 \partial q_1} \frac{dq_1}{dE} + \frac{\partial^2 u}{\partial q_1 \partial q_2} \frac{dq_2}{dE} + \cdots \frac{\partial^2 u}{\partial q_1 \partial q_n} \frac{dq_n}{dE} - p_1 \frac{d\mu}{dE} = 0$$

which following our derivations in Equations 2.6 and 2.9 can be written in matrix form as

$$(2.12) \quad \begin{bmatrix} \frac{\partial^2 u}{\partial q_1 \partial q_1} & \frac{\partial^2 u}{\partial q_1 \partial q_2} & \cdots & \frac{\partial^2 u}{\partial q_1 \partial q_n} \end{bmatrix} \begin{bmatrix} \frac{dq_1}{dE} \\ \frac{dq_2}{dE} \\ \vdots \\ \frac{dq_n}{dE} \end{bmatrix}$$

$$U \frac{dq}{dE} - p' \frac{d\mu}{dE} = 0$$

Joining these two results with the differentiation of the budget constraint yields Barten's fundamental matrix

$$(2.13) \quad \begin{bmatrix} U & p \\ p' & 0 \end{bmatrix} \begin{bmatrix} \frac{dq}{dE} & \frac{dq}{dp'} \\ -\frac{d\mu}{dE} & -\frac{d\mu}{dp'} \end{bmatrix} = \begin{bmatrix} 0 & \mu I \\ 1 & -q' \end{bmatrix}$$

3. MATRIX CALCULUS

For more detail on the individual steps in this paper see Theil [2]. Several of the steps could be simplified using the material presented in Magnus and Neudecker [1].

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1175 MCCARTY HALL,
P.O. Box 110240
GAINESVILLE, FL 32611-0240
E-mail address, Charles B. Moss: cbmoss@ufl.edu
URL: <http://ricardo.ifas.ufl.edu/>