

Black Scholes: Lecture XXXVIII

Charles B. Moss

November 19, 2014

1 Black-Scholes Option Pricing

2 Ito's Lemma Formulation

Black-Scholes Option Pricing

- In Lecture 35 we developed the distribution for the return on the stock and future consumption

$$\begin{pmatrix} \ln(x_j/S_j) \\ \rho \ln(C/C_0) \end{pmatrix} \sim N \left(\begin{bmatrix} \mu_j - \ln(S_j) \\ -B\mu_C + \ln(\rho) + B \ln(C_0) \end{bmatrix}, \begin{bmatrix} \sigma_j^2 & -B\kappa\sigma_j\sigma_C \\ -B\kappa\sigma_j\sigma_C & B^2\sigma_C^2 \end{bmatrix} \right) \quad (1)$$

- In this formulation $\ln(x_j/S_j)$ is the gain from a short sale (i.e. if I sell the stock in this period $-\ln(S_j)$ to buy it back in the second part of the analysis $\ln(x_j)$.
 - S_j is the stock price today and x_j is the stock price tomorrow.
 - The expected return from this sale is $\mu_j - \ln(S_j)$ where μ_j is the expected value of the stock at the end of the holding period.

- Continued

- $\rho \ln(C/C_0)$ is the present value of consumption tomorrow.
 - B is the risk aversion coefficient from the power utility function.
 - κ is the correlation coefficient between consumption and the stock price in the future.
- The value of the call option can then be written as

$$p_j(S_j, k) = S_j \int_{-\infty}^{\infty} \int_{\ln(k/S_j)}^{\infty} \left(e^z - \frac{k}{S_j} \right) e^y f(z, y) dz dy \quad (2)$$

- Continued

- Remember that k is the strike price for the call option (the price that you can buy the stock for under the option).
- k/S_j is then the strike price as a percent of the option value. Compare this with e^z at the lower bound of the inner integral, the value inside the parenthesis is equal to zero. For any z above this the value in the inner parenthesis is positive (the option pays off).

- Distributing the integral over the inner parenthesis

$$\begin{aligned}
 p_j(S_j, k) = & S_j \int_{-\infty}^{\infty} \int_{\ln(k/S_j)}^{\infty} e^{z+y} f(z, y) dz dy \\
 & - k \int_{-\infty}^{\infty} \int_{\ln(k/S_j)}^{\infty} e^y f(z, y) dz dy
 \end{aligned} \tag{3}$$

- Some mathematical niceities
 - Dealing with the lower bound of the integral

$$\begin{aligned}
 \int_a^\infty f(z) dz &\Rightarrow v = \frac{a - \mu}{\sigma} \Rightarrow \int_a^\infty f(z) dz = \int_{\frac{a - \mu}{\sigma}}^\infty n(v) dv \\
 \int_{\frac{a - \mu}{\sigma}}^\infty n(v) dv &= \int_{-\infty}^{-\frac{a - \mu}{\sigma}} n(v) dv \\
 &= N\left(\frac{-1 + \mu}{\sigma}\right)
 \end{aligned}
 \tag{4}$$

Thus, the integral has been transformed to the standard normal cumulative density function.

- Next, because of the geometric nature of the distribution

$$\begin{aligned}
 \int_a^\infty e^z f(z) dz &= \int_a^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp \left[-\frac{1}{2\sigma^2} (z - \mu)^2 + z \right] dz \\
 &= \left(e^{\mu + \sigma^2/2} \right) \int_a^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp \left\{ -\frac{1}{2\sigma^2} (z - [\mu - \sigma^2])^2 \right\} dz \\
 &= \left(e^{\mu + \sigma^2/2} \right) N \left(\frac{-a + \mu}{\sigma} + \sigma \right)
 \end{aligned} \tag{5}$$

- To simplify the integration, we consider the conditional distribution

$$\int_{-\infty}^{\infty} e^y f(y|z) dy = \exp \left[\mu_c + \kappa \frac{\sigma_c}{\sigma_j} (z - \mu_j) + \frac{1}{2} (1 - \kappa^2) \sigma_c^2 \right] \quad (6)$$

The second double integral in Equation 3 then become

$$\begin{aligned} \int_{-\infty}^{\infty} \int_a^{\infty} e^y f(z) dz dy &= \int_a^{\infty} f(z) \int_{-\infty}^{\infty} e^y f(y|z) dy dz \\ &= \int_a^{\infty} e^{\mu_c + \kappa \frac{\sigma_c}{\sigma_j} (z - \mu_j) + (1 - \kappa^2) \frac{\sigma_c^2}{2}} f(z) dz \\ &= \left(e^{\mu_j + \mu_c + [\sigma_j^2 + 2\kappa\sigma_j\sigma_c + \sigma_c^2]/2} \right) N \left(\frac{-a + \mu_j}{\sigma_j} + \kappa\sigma_c + \sigma_j \right) \quad (7) \end{aligned}$$

- Thus given

$$\int_{-\infty}^{\infty} e^y f(z, y) dz dy = \left(e^{\mu_c + \sigma_c^2/2} \right) N \left(\frac{-a + \mu_j}{\sigma_j} + \kappa \sigma_C \right) \quad (8)$$

by Equation 5, and

$$\begin{aligned} \int_{-\infty}^{\infty} \int_a^{\infty} e^{z+y} f(z, y) dz dy &= \left(e^{\mu_j + \mu_c + [\sigma_j^2 + 2\kappa \sigma_j \sigma_C + \sigma_C^2]/2} \right) \\ &\times N \left(\frac{-a + \mu_j}{\sigma_j} + \kappa \sigma_C + \sigma_j \right) \end{aligned} \quad (9)$$

- We have that

$$\begin{aligned}
 p_j(S_j, k) &= S_j \int_{-\infty}^{\infty} \int_{\ln(k/S_j)}^{\infty} e^{z+y} f(z, y) dz dy \\
 &\quad - k \int_{-\infty}^{\infty} \int_{\ln(k/S_j)}^{\infty} e^y f(z, y) dz dy \\
 &= S_j \left(e^{\mu_j + \mu_c + [\sigma_j^2 + 2\kappa\sigma_j\sigma_c + \sigma_c^2]/2} \right) N\left(\frac{-a + \mu_j}{\sigma_j} + \kappa\sigma_c + \sigma_j\right) - \\
 &\quad k \left(e^{\mu_c + \sigma_c^2/2} \right) N\left(\frac{-a + \mu_j}{\sigma_j} + \kappa\sigma_c\right)
 \end{aligned} \tag{10}$$

- To finish the derivation, we assume

$$e^{\mu_c + \frac{1}{2}\sigma_c^2} = \frac{1}{1 + r_f} \tag{11}$$

- or that future consumption is discounted at the risk-free rate of return. In addition, we assume that

$$e^{\mu_j + \mu_C + [\sigma_j^2 + 2\kappa\sigma_j\sigma_C + \sigma_C^2]/2} = 1 \quad (12)$$

which is implicitly the pricing condition of stock in period 0 given its utility distribution in period 1 (enforces a zero arbitrage condition on the stock price).

- The Black-Scholes formula is then

$$Z \equiv \frac{\ln\left(\frac{S_j}{k}\right) + (\mu_j + \kappa\sigma_j\sigma_C)}{\sigma_j} \quad (13)$$

$$p_j(S_j, k) = S_j N(Z + \sigma_j) - \frac{1}{1 + r_f} k N(Z)$$

Ito's Lemma formulation

- Stochastic Process: the equation of motion

$$\Delta E = \mu \Delta t + \sigma \Delta z \quad (14)$$

- Defining the Wiener increment (following Kamien and Schwartzs definitions):

$$\Delta z = \epsilon_t \sqrt{\Delta t} \quad (15)$$

- The expectation and the variance of the equation of motion for equity can then be defined as

$$\begin{aligned} E[\Delta E] &= \mu \Delta t \\ V[\Delta E] &= \sigma^2 \Delta t \end{aligned} \quad (16)$$

- Thus, we can rewrite the Black and Scholes result using stochastic process as

$$\begin{aligned} c &= SN(d_1) - Xe^{-r_f T} N(d_2) \\ d_1 &= \frac{\ln\left(\frac{S}{X}\right) + r_f T}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned} \quad (17)$$

Table : Black-Scholes Example - One Week

Strike Price	Price Today	Weeks to Mat.	Std. Dev	T	d_1	d_2	Option Price
87	89	1	0.15	0.019	1.126	1.105	2.164
88	89	1	0.15	0.019	0.577	0.556	1.37
89	89	1	0.15	0.019	0.034	0.013	0.76
90	89	1	0.15	0.019	-0.504	-0.524	0.36
91	89	1	0.15	0.019	-1.035	-1.056	0.143

Table : Black-Scholes Example - Ten Weeks

Strike Price	Price Today	Weeks to Mat.	Std. Dev	T	d_1	d_2	Option Price
87	89	10	0.15	0.192	0.452	0.386	3.711
88	89	10	0.15	0.192	0.278	0.212	3.097
89	89	10	0.15	0.192	0.106	0.04	2.549
90	89	10	0.15	0.192	-0.064	-0.13	2.069
91	89	10	0.15	0.192	-0.232	-0.298	1.656

Figure : Comparison of Option Values

