

Introduction to Options and Futures: Lecture XXXVII

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1 Futures and the Hedge

2 Options

3 Black-Scholes

Futures and the Hedge

- Futures markets such as the Chicago Board of Trade allow for the trading (purchase and sale) of commodities to be delivered at some future date. On November 18, 2004 the future price for December 2004 corn was 204'0.
 - From a risk management standpoint farmers/decision makers could choose to forward market cattle in November on August 2, 2004.
 - Forward marketing using futures markets is typically referred to as hedging. To forward market cattle on August 2, 2004 for sale on November 18, 2004 the farmer would sell a contract of fat cattle on August 2.

Table : Futures Prices for Fat Cattle

Date	Open	High	Low	Close
11/17/2004	88.05	89.10	87.95	88.90
8/2/2004	89.30	90.02	89.25	89.62
6/17/2004	88.45	88.75	88.15	88.60
<i>Transactions</i>				
Cash Price	85.44	Sell on 8/2		89.62
		Buy on 11/17		88.90

- Continued

- When the farmer markets cattle on November 18, 2004 he receives \$85.44/cwt on the cash market.
- In addition, he has gained \$0.72/cwt on the futures market (the price that he buys back the futures contract for is \$88.90/cwt).
- If the farmer had attempted to market his cattle on June 17, 2004 instead the farmer would loose \$0.30/cwt on each contract resulting in a \$85.14/cwt price.

- In general, the return on a hedge is based on the expected basis, which is defined as the difference between the cash price and the futures price on the date that the futures transaction is reversed.

$$B_t = F_t - C_t \quad (1)$$

- Rearranging this expression

$$C_t = F_t - B_t \quad (2)$$

or taking the expectation

$$E(C_t) = E(F_t) - E(B_t) \quad (3)$$

- So the expected cash price at the date the hedge is initiated (on August 2, 2004) less the expected basis $E[B_t]$. Any risk in the price then comes from the risk of the basis.

Options

- What is an option?
 - In a general sense, an option is exactly what its name implies. An option is the opportunity to buy or sell one share of stock or lot of commodity at some point in the future at some stated price.

Table : Price of Fat Cattle Calls for December

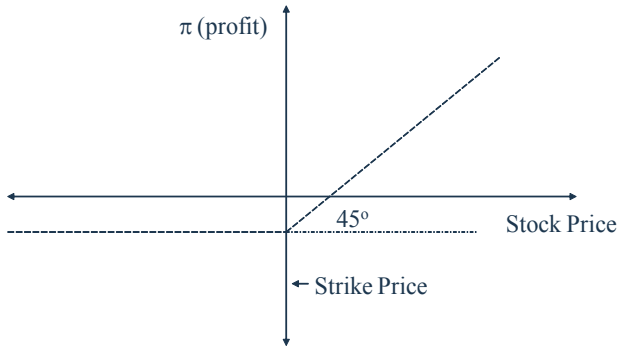
Strike	Open	High	Low	Last	Settle	Change
86	3.400	3.400	3.400	3.400	3.525	-1650
88	2.500	2.750	2.500	2.700	2.600	-1300
89	2.400	2.400	2.400	2.400	2.200	-1200
90	1.600	1.900	1.600	1.600	1.700	-1100

- Continued

- From Table 2, the right to purchase fat cattle at \$88.00/cwt in December cost \$2.60/cwt.
- On the flip side, the instrument that gives the bearer the right to sell a stock or unit of commodity at a stated price is called a put.

- Options are contingent assets they only have a value contingent on certain outcomes in the economy.
 - If you purchase a call with a strike price of \$88.00/cwt for \$2.60 and the cattle price increases to \$92.00/cwt next month, you would exercise the option grossing \$4.00/cwt ($\$92.00/\text{cwt} - \$88.00/\text{cwt}$). The net return is \$1.40/cwt ($\$4.00/\text{cwt} - \$2.60/\text{cwt}$).
 - If the price of cattle decreases to \$84.00/cwt in December the option becomes valueless. Exercising the option would result in a \$4.00/cwt gross loss ($\$84.00/\text{cwt} - \$88.00/\text{cwt}$) and a total loss of \$6.60 ($\$4.00/\text{cwt}$ from exercising the option and \$2.60/cwt on the purchase price of the option).

Figure : Graph of Call Profit



- Technically, there are two types of options: a European option and an American option.
 - A European option can only be exercised on the expiration date.
 - The American option can be exercised on any date up until the expiration date.
 - $F(S, t; T, x)$ denote the value of an American call option with stock price S on date t with expiration date of T for an exercise price of x ,
 - $f(S, t; T, x)$ is the European call option,
 - $G(S, t; T, x)$ is an American put option, and
 - $g(S, t; T, x)$ is the European put option.

- Propositions

- Proposition 1: $F(.) \geq 0$, $G(.) \geq 0$, $f(.) \geq 0$, $g(.) \geq 0$.
- Proposition 2

$$\begin{aligned} F(S, T; T, x) &= f(S, T; T, x) = \max(S - x, 0) \\ G(S, T; T, x) &= g(S, T; T, x) = \max(x - S, 0) \end{aligned} \quad (4)$$

- Proposition 3

$$\begin{aligned} F(S, t; T, x) &\geq S - X \\ G(S, t; T, x) &\geq x - S \end{aligned} \quad (5)$$

- Continued

- Proposition 4: For $T_2 > T_1$

$$\begin{aligned} F(., T_2, x) &\geq F(., T_1, x) \\ G(., T_2, x) &\geq G(., T_1, x) \end{aligned} \tag{6}$$

- Proposition 5: $F(.) \geq f(.)$ and $G(.) \geq g(.)$.
- Proposition 6: For $x_1 > x_2$

$$\begin{aligned} F(., x_1) &\leq F(., x_2) \text{ and } f(., x_1) \leq f(., x_2) \\ G(., x_1) &\geq G(., x_2) \text{ and } g(., x_1) \geq g(., x_2) \end{aligned} \tag{7}$$

- Continued

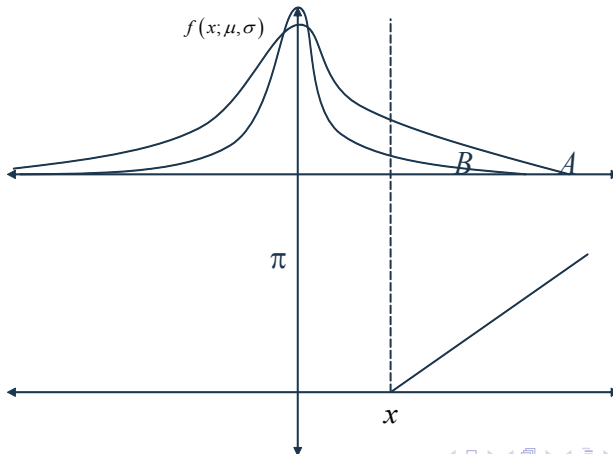
- Proposition 7:

$$S = F(S, t; \infty, 0) \geq F(S, t; T, x) \geq f(S, t; T, x)$$

- Proposition 8: $f(0, \cdot) = F(0, \cdot) = 0$

- Three of the previous results bear restating:
 - The value of a call option is an increasing function of the spot stock price (S).
 - The value of a call option is a decreasing price of the strike price (X).
 - The value of a call option is an increasing function of the time to maturity (T).
- The value of an option is an increasing function of the variability of the underlying asset. To see this, think about imposing the probability density function over a "zero price" option.

Figure : Valuing the Call Option



Black-Scholes

- The European Call Option
 - First, we construct the payoff function for a security on which the option is written as

$$x_j(k) = \begin{cases} x_j - k & \text{if } x_j - k > 0 \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

where $x_j(k)$ is the payoff a share of security j purchased and an exercise price of k .

- The price of the European call option is then defined as

$$p_j(S_j, k) \geq \max \left[S_j - \frac{k}{1 + r_f}, 0 \right] \quad (9)$$

- Consider the strategy of selling one share of the security to buy one European call option written on the security.
 - The initial cost of the strategy is

$$p_j(S_j, k) - S_j + \frac{k}{1 + r_f} \quad (10)$$

- The possible payoffs of this strategy are

$$\left\{ \begin{array}{l} x_j - k - x_j + k = 0 \text{ if } x_j \geq k \\ -x_j + k > 0 \text{ if } x_j < k \end{array} \right\} \quad (11)$$

- Continued
- In the first case, you exercise the option buying back the stock at the original price while in the second case the investor makes money because the stock price decreased (you make profit equal to the decrease in the stock price).
- Therefore, to avoid a risk-less profit (you can't make something for nothing)

$$\begin{aligned}
 p_j(S_j, k) - S_j + \frac{k}{1 + r_f} &> 0 \\
 \Rightarrow p_j(S_j, k) &> S_j - \frac{k}{r_f}
 \end{aligned}
 \tag{12}$$

- Starting with a two-period economy, we start by assuming a power utility function for the representative agent

$$u_0(z_0) + u_1(z_1) = \frac{1}{1-B} z_0^{1-B} + \rho \frac{1}{1-B} z_1^{1-B} \quad (13)$$

where ρ is the time preference parameter.

- The arbitrage condition (selling short a share of stock and purchasing a call option) then implies

$$p_j(S, k) = \rho E \left[\max[x_j - k, 0] \left(\frac{C}{C_0} \right)^{-B} \right] \quad (14)$$

- Next, we assume that x and C are lognormally distributed

$$\begin{pmatrix} \ln(x_j) \\ \ln(C) \end{pmatrix} \sim N \left(\begin{bmatrix} \mu_j \\ \mu_C \end{bmatrix}, \begin{bmatrix} \sigma_j^2 & \kappa \sigma_j \sigma_C \\ \kappa \sigma_j \sigma_C & \sigma_C^2 \end{bmatrix} \right) \quad (15)$$

where κ is the correlation coefficient.

- This assumption implies that $\ln(S/x)$ (the return on the short sale) and $\rho \ln(C/C_0)$ are normally distributed

$$\begin{pmatrix} \ln(x_j/S_j) \\ \rho \ln(C/C_0) \end{pmatrix} \sim N \left(\begin{bmatrix} \mu_j - \ln(S_j) \\ -B\mu_C + \ln(\rho) + B \ln(C_0) \end{bmatrix}, \begin{bmatrix} \sigma_j^2 & -B\kappa\sigma_j\sigma_C \\ -B\kappa\sigma_j\sigma_C & B^2\sigma_C^2 \end{bmatrix} \right) \quad (16)$$

- The value of the call option can then be written as

$$p_j(S_j, k) = S_j \int_{-\infty}^{\infty} \int_{\ln(k/S_j)}^{\infty} \left(e^z - \frac{k}{S_j} \right) e^y f(z, y) dz dy \quad (17)$$