

# The Arbitrage Pricing Model: Lecture XXXVI

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## 1 Empirical Applications of Capital Market Models

- Sharpe-Lintner CAPM
- Black's Model
- Tests for CAPM Efficiency
- Cross-Sectional Regression
- Two-Tests
- Arbitrage Pricing Model

# Empirical Applications of Capital Market Models

- The basic Sharpe-Lintner CAPM Model

$$\begin{aligned} E[R_i] &= R_f + \beta_{im} (E[R_m] - R_f) \\ \beta_{im} &= \frac{\text{Cov}[R_i, R_m]}{\text{Var}[R_m]} \end{aligned} \quad (1)$$

Assuming that the riskfree asset exists

$$\left. \begin{aligned} E[R_i] - R_f &= \beta_{im} (E[R_m] - R_f) \\ Z_i &= E[R_i] - R_f \\ Z_m &= E[R_m] - R_f \end{aligned} \right\} \Rightarrow Z_i = \alpha_i + \beta_i Z_m + \epsilon_i \quad (2)$$

- Constructing a dataset of 43 stocks from the Center for Research into Security Prices (CRSP) dataset, using the return on the Standard and Poors 500 portfolio and using the 3 month treasury bill as the market portfolio the estimated parameters for the CAPM model are presented in Table 1.

Table : CAPM Parameters from the Center for Research into Security Prices Dataset

| Parameter  | 11499                | 12140                 | 15553                | 18542                | 19166                | 19749                 |
|------------|----------------------|-----------------------|----------------------|----------------------|----------------------|-----------------------|
| $\alpha_i$ | 0.00651<br>(0.01459) | -0.00267<br>(0.00522) | 0.00757<br>(0.00324) | 0.00223<br>(0.00494) | 0.00092<br>(0.00539) | -0.00465<br>(0.01091) |
| $\beta_i$  | 1.03884<br>(0.31641) | 1.3945<br>(0.11317)   | 0.30513<br>(0.07028) | 0.99853<br>(0.10708) | 1.08365<br>(0.11686) | -0.00807<br>(0.23661) |

## Black's Model

- An alternative to the model presented by Sharpe and Lintner is the zero-beta model suggested by Black

$$E[R_i] = E[R_{0m}] + \beta_{im} (E[R_m] - E[R_{0m}]) \quad (3)$$

where  $R_{0m}$  is the return on the zero-beta portfolio, or the minimum variance portfolio that is uncorrelated with the market portfolio.

Table : Black Parameters from the Center for Research into Security Prices Dataset

| Parameter  | 11499                | 12140                 | 15553               | 18542                | 19166                | 19749                 |
|------------|----------------------|-----------------------|---------------------|----------------------|----------------------|-----------------------|
| $\alpha_i$ | 0.00636<br>(0.01485) | -0.00422<br>(0.00531) | 0.01033<br>(0.0033) | 0.00236<br>(0.00502) | 0.00056<br>(0.00548) | -0.00057<br>(0.0111)  |
| $\beta_i$  | 1.03803<br>(0.31591) | 1.39097<br>(0.11303)  | 0.3089<br>(0.07028) | 0.98751<br>(0.10691) | 1.08609<br>(0.11667) | -0.00944<br>(0.23618) |

Table : Comparison of Sharpe-Lintner to Black

|       | Sharpe-Lintner | Black    |
|-------|----------------|----------|
| 11499 | 1.03884        | 1.03803  |
| 12140 | 1.3945         | 1.39097  |
| 15553 | 0.30513        | 0.3089   |
| 18542 | 0.99853        | 0.98751  |
| 19166 | 1.08365        | 1.08609  |
| 19749 | -0.00807       | -0.00944 |

# Sharpe-Lintner Model

- Estimated parameters

$$\hat{\alpha} = \hat{\mu} - \hat{\beta}\hat{\mu}_m \quad \hat{\mu} = \frac{1}{T} \sum_{t=1}^T R_t \quad \hat{\mu}_m = \frac{1}{T} \sum_{t=1}^T Z_{mt}$$

$$\hat{\beta} = \frac{\sum_{t=1}^T (Z_t - \hat{\mu})(Z_{mt} - \hat{\mu}_m)}{\sum_{t=1}^T (Z_{mt} - \hat{\mu}_m)^2} \quad (4)$$

$$\hat{\Sigma} = \frac{1}{T} \sum_{t=1}^T \left( Z_t - \hat{\alpha} - \hat{\beta}Z_{mt} \right) \left( Z_t - \hat{\alpha} - \hat{\beta}Z_{mt} \right)$$

- Test Statistics for efficiency

$$\hat{\alpha} \sim N \left( \alpha, \frac{1}{T} \left[ 1 + \frac{\hat{\mu}_m^2}{\sigma_m^2} \right] \Sigma \right) \quad (5)$$

$$\hat{\beta} \sim N \left( \beta, \frac{1}{T} \left[ \frac{1}{\sigma_m^2} \right] \Sigma \right)$$

Hypothesis  $H_0 : \alpha = 0$  versus  $H_1 : \alpha \neq 0$

$$J_0 = T \left[ 1 + \frac{\mu_m^2}{\sigma_m^2} \right]^{-1} \hat{\alpha}' \Sigma \hat{\alpha} \quad (6)$$

- Fama, E. and J. MacBeth. 1973. Risk, Returns, and Equilibrium: Empirical Tests *Journal of Political Economy* 81, 607-636.
- Using either set of betas, the question then becomes whether the expected returns are consistent with their betas

$$E[Z_i] = \gamma_0 + \gamma_1 \beta_i \quad (7)$$

Table : Cross Sectional Estimates

| Parameter  | Estimate               |
|------------|------------------------|
| $\gamma_0$ | 0.006062<br>(0.000592) |
| $\gamma_1$ | 0.906548<br>(0.092863) |

- Two Tests

- Sharpe-Lintner test of the constant

$$\gamma_0 = 0 \Rightarrow \frac{0.006062}{0.000592} = 10.23986 \quad (8)$$

- Beta explain the variations in expected returns

$$\gamma_1 = 1 \Rightarrow \frac{1.0 - 0.906548}{0.092863} = 1.006343 \quad (9)$$

- A little reformulation

$$E[Z_i] = \gamma_0 + \gamma_1 \beta_i + \gamma_2 D_i \quad (10)$$

where  $D_i$  is a dummy variable which is equal to one if the stock is an agribusiness stock and zero otherwise.

Table : Security Market Line Regression

| Parameter  | Estimate               |
|------------|------------------------|
| $\gamma_0$ | 0.006083<br>(0.000605) |
| $\gamma_1$ | 0.913067<br>(0.097456) |
| $\gamma_2$ | -0.00035<br>(0.001374) |

# Arbitrage Pricing Model

- As we discussed in previous lectures, the returns in the arbitrage pricing model are assumed to be determined by a linear factor model

$$R_t = a + bf_t + \epsilon_t \quad (11)$$
$$R_t \in M_{N \times 1}, f_t \in M_{k \times 1}, b \in M_{N \times k}$$

- $R_t$  is a vector of  $N$  asset returns.
- $f_t$  is a vector of  $k$  common factors.
- $b$  is a  $N \times k$  matrix of factor loadings.

- The arbitrage pricing equilibrium implies that the expected return on the vector of assets is a linear function of the factor loadings

$$E[R] = \lambda_0 + b\lambda' \quad (12)$$

- Two ways to define the common factors: (1) Endogenously based on returns, or (2) Exogenously based on macroeconomic variables.

- Given the linear factor model above, the variance matrix for the returns on the vector of assets becomes

$$\Sigma = \beta\beta' + \Theta \quad (13)$$

where  $\Theta$  is a diagonal matrix. Under this specification, we can estimate the vector of factor loadings by maximizing

$$L = -\ln |\beta\beta' + \Theta| - \text{tr} \left[ S (\beta\beta' + \Theta)^{-1} \right] \quad (14)$$

Table : Factor Loadings

| Stock Number | Factor 1 | Factor 2 |
|--------------|----------|----------|
| 11499        | 0.00576  | -0.05389 |
| 12140        | 0.03512  | -0.06152 |
| 15553        | 0.02569  | -0.00286 |
| 18542        | 0.01408  | -0.05615 |
| 19166        | 0.01922  | -0.0578  |
| 19749        | -0.01091 | -0.00215 |
| 11499        | 0.00576  | -0.05389 |

Table : Estimated Risk Premia

| Parameter   | Estimate              |
|-------------|-----------------------|
| $\lambda_0$ | 0.01123<br>(0.00200)  |
| $\lambda_1$ | 0.00397<br>(0.05674)  |
| $\lambda_2$ | -0.07589<br>(0.03665) |

- Augmenting the model to test for disequilibria in the equity market for Agribusinesses

$$E[R_i] = \lambda_0 + \lambda_1 b_{1i} + \lambda_2 b_{2i} + \lambda_3 D_i \quad (15)$$

Table : APM with Agribusiness Coefficient

| Parameter   | Estimate              |
|-------------|-----------------------|
| $\lambda_0$ | 0.01061<br>(0.00198)  |
| $\lambda_1$ | -0.0094<br>(0.05568)  |
| $\lambda_2$ | -0.08497<br>(0.03601) |
| $\lambda_3$ | 0.00413<br>(0.00229)  |