

The Arbitrage Pricing Model: Lecture XXXV

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1 Single Factor Model

2 Multifactor Models

A Single Factor Model

- Abstracting away from the specific form of the CAPM model, we posit a single factor model written as

$$\tilde{z}_i = a_i + \sum_{k=1}^K b_{ik} \tilde{f}_k + \tilde{\epsilon}_i \quad (1)$$

- In this model, the random return on an investment $\tilde{z}_i$ is a linear function of some random factor $\tilde{f}_i$ and an idiosyncratic term $\tilde{\epsilon}_i$.

$$\begin{aligned} E(\tilde{\epsilon}_i) &= E(\tilde{f}_k) = E(\tilde{\epsilon}_i \tilde{f}_k) = E(\tilde{f}_k \tilde{f}_l) = 0 \\ E(\tilde{\epsilon}_i^2) &= s_i^2 < S_i^2 \\ E(\tilde{f}_k^2) &= 1 \end{aligned} \quad (2)$$

- Abstracting away from the idiosyncratic risk

$$\tilde{z}_i = a_i + b_i \tilde{f}_i \quad (3)$$

- If the b_i s of two assets are the same, then the a_i s must be the same for an arbitrage free model.
- Suppose we are interested in forming a portfolio of two assets with different b_i s, $b_i \neq b_j$, $b_i \neq 0$, $b_j \neq 0$

$$\begin{aligned} z &= w(a_i + b_i f) + (1 - w)(a_j + b_j f) \\ &= wa_i + wb_i f + a_j - wa_j + b_j f - wb_j f \\ &= [w(a_i - a_j) + a_j] + [w(b_i - b_j) + b_j] f \end{aligned} \quad (4)$$

- Continued
 - Computing the mean and variance of this portfolio yields

$$\begin{aligned}
 E[z] &= w(a_i - a_j) + a_j \\
 V[z] &= E([w(a_i - a_j) + a_j] + [w(b_i - b_j) + b_j]f)^2 \\
 &\quad - (w(a_i - a_j) + a_j)^2 \\
 &= E(w(a_i - a_j) + a_j)^2 + E\{[w(a_i - a_j) + a_j][w(b_i - b_j) + b_j]f\} \\
 &\quad + E([w(b_i - b_j) + b_j]f^2) - (w(a_i - a_j) + a_j)^2 \\
 &\quad = [w(b_i - b_j) + b_j]^2
 \end{aligned}
 \tag{5}$$

- Holding the variance of the portfolio equal to zero, we find

$$\begin{aligned} [w(b_i - b_j) + b_j]^2 &= 0 \\ w(b_i - b_j) + b_j &= 0 \\ w^* &= \frac{b_j}{b_j - b_i} \end{aligned} \tag{6}$$

- Solving for z the portfolio return yields

$$\begin{aligned} z &= \left[\frac{b_j}{b_j - b_i} \right] (a_i - a_j) + a_j + \left[\left(\frac{b_j}{b_j - b_i} \right) (b_i - b_j) + b_j \right] f \\ &= \frac{b_j (a_i - a_j)}{b_j - b_i} + a_j = R \text{ the riskfree rate} \\ \frac{(a_i - a_j)}{(b_j - b_i)} &= \frac{R - a_j}{b_j} = \frac{R - a_i}{b_i} \end{aligned} \tag{7}$$

- Continued
 - Extending the results

$$\frac{R - \bar{z}_i}{b_j} = \frac{(a_i - a_j)}{(b_j - b_i)}$$

$$R - \bar{z}_i = \frac{(a_i - a_j)}{(b_j - b_i)} b_j$$

$$R - \frac{(a_i - a_j)}{(b_j - b_i)} b_j = \bar{z}_i \tag{8}$$

$$\bar{z}_i = R + \frac{(a_i - a_j)}{(b_j - b_i)} b_j \Rightarrow \lambda_1 = \frac{(a_i - a_j)}{(b_j - b_i)}$$

$$\bar{z}_i = R + \lambda_1 b_i$$

$$\bar{z}_i = \lambda_0 + \lambda_1 b_i$$

Multifactor Models

- Suppose that asset returns are generated by a two factor linear model

$$\tilde{z}_i = a_i + b_i \tilde{f}_1 + c_i \tilde{f}_2 \quad (9)$$

- A portfolio of these assets then yields

$$\sum_i w_i \tilde{z}_i = \sum_i w_i a_i + \sum_i w_i b_i \tilde{f}_1 + \sum_i w_i c_i \tilde{f}_2 \quad (10)$$

- Again to minimize the systematic risk

$$\sum_i w_i b_i = \sum_i w_i c_i = 0 \quad (11)$$

- If the portfolio is riskless, then it yields zero profit

$$R = \sum_i w_i a_i$$

$$\sum_i w_i a_i - R = 0 \quad (12)$$

- Given $\sum_i w_i = 1 \Rightarrow \sum_i w_i R = R$

$$\therefore \sum_i w_i (a_i - R) = 0$$

$$\Rightarrow \begin{pmatrix} a_1 - R & a_2 - R & a_3 - R \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (13)$$

- The matrix

$$\begin{pmatrix} a_1 - R & a_2 - R & a_3 - R \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \quad (14)$$

must be singular, or the first row must be a linear combination of the last two rows

$$\begin{pmatrix} a_1 - R & a_2 - R & a_3 - R \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \Rightarrow R_1 + \lambda_1 R_2 + \lambda_2 R_3 = 0$$

$$\bar{z}_i - R = a_i - R = \lambda_1 b_i + \lambda_2 c_i \forall i \text{ or}$$

$$\bar{z}_i = \lambda_0 + \lambda_1 b_i + \lambda_2 c_i \forall i$$

(15)