

Capital Asset Pricing Model and the Arbitrage Pricing Theorem : Lecture XXXIV

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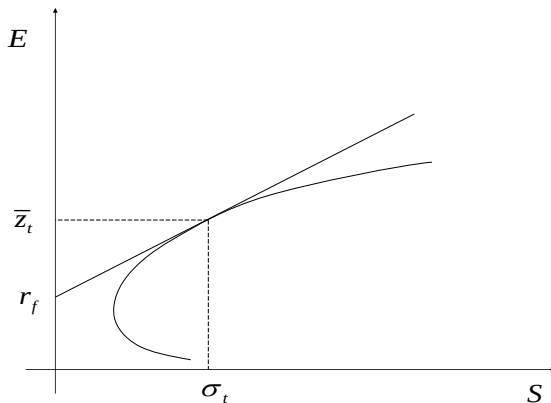
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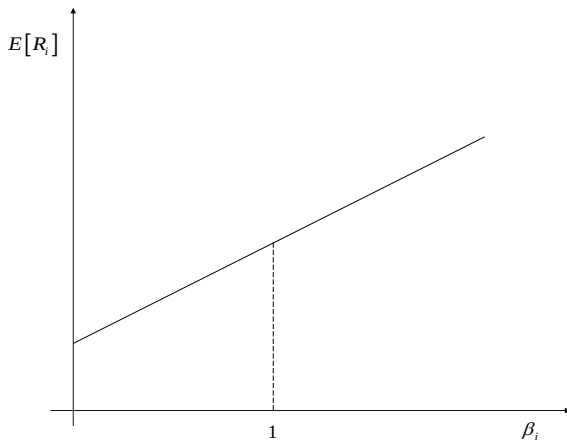
Capital Asset Pricing Model

- Start by returning to the efficient Expected Value - Standard Deviation frontier presented in Figure 4.
- Using the results from the Expected Value-Standard Deviation frontier, we can derive the security market line (SML) as depicted in Figure 5.

Capital Market Line: Efficient Expected Value – Standard Deviation Frontier



Security Market Line



- Which is consistent with the standard CAPM relationship

$$E[r_j] = \bar{r} + \beta [E(r_m) - \bar{r}] \quad (1)$$

- Starting with a two-asset portfolio, we construct a portfolio using market portfolio (m) and asset i .

$$\begin{aligned} E[R_p] &= \alpha E[R_i] + (1 - \alpha) E[R_m] \\ \sigma_p(R_p) &= \left[\alpha^2 \sigma_i^2 + 2(1 - \alpha) \alpha \sigma_{im} + (1 - \alpha)^2 \sigma_m^2 \right]^{1/2} \end{aligned} \quad (2)$$

- Next, we examine the risk/return relationship based on changes in the share of asset

$$\begin{aligned} \frac{\partial E[R_p]}{\partial \alpha} &= E[R_i] - E[R_m] \\ \frac{\partial \sigma(R_p)}{\partial \alpha} &= \frac{1}{2} \left[\alpha^2 \sigma_i^2 + 2(1-\alpha)\alpha\sigma_{im} + (1-\alpha)^2 \sigma_m^2 \right]^{-1/2} \\ &\quad \times \left[2\alpha\sigma_i^2 - 2\sigma_m^2 + 2\alpha\sigma_m^2 + 2\sigma_{im} - 4\alpha\sigma_{im} \right] \end{aligned} \quad (3)$$

- Consider what happens as the share held in asset becomes small

$$\begin{aligned} \left. \frac{\partial \sigma(R_p)}{\partial \alpha} \right|_{\alpha \rightarrow 0} &= \frac{\sigma_{im} - \sigma_m^2}{\sigma_m} \\ &= \left[\frac{\sigma_{im} - \sigma_m^2}{\sigma_m} \right] \frac{\sigma_m}{\sigma_m} = [\beta - 1] \sigma_m \end{aligned} \quad (4)$$

- The risk/return relationship as the share in asset becomes small is then computed as

$$\therefore \left. \frac{\frac{\partial E[R_p]}{\partial \alpha}}{\frac{\partial \sigma(R_p)}{\partial \alpha}} \right|_{\alpha \rightarrow 0} = \frac{E[R_i] - E[R_m]}{[\beta - 1] \sigma_m} \quad (5)$$

$$\frac{1}{\beta - 1} \frac{E[R_i] - E[R_m]}{\sigma_m}$$

- This relationship then yields $(E[R_m] - r_f) / \sigma_m$

$$\begin{aligned} \therefore \frac{1}{\beta - 1} \frac{E[R_i] - E[R_m]}{\sigma_m} &= \frac{E[R_m] - r_f}{\sigma_m} \\ E[R_i] - E[R_m] &= (\beta - 1) [E[R_m] - r_f] \\ &= \beta [E[R_m] - r_f] - E[R_m] + r_f \\ \Rightarrow E[R_i] - r_f &= \beta [E[R_m] - r_f] \end{aligned} \quad (6)$$

Empirical Tests of CAPM

- The typical estimation procedure for empirically testing the CAPM is a two step model.
 - First, the annual returns are estimated as a function of the returns on the market portfolio

$$R_{jt} = \alpha_j + \beta_j R_{mt} \quad (7)$$

- Using the estimated results from the first estimation, the SML is estimated across equations

$$\bar{R}_j = \hat{\gamma}_0 + \hat{\gamma}_1 \beta_j + u_j \quad (8)$$

- The "testable" implications of the CAPM are:
 - The intercept term γ_0 should be equal to zero.
 - β should be the only factor that explains the rate of return on a risky asset.
 - The relationship should be linear.
 - The coefficient γ_1 should be equal to $\bar{R}_{mt} - R_f$.

Uses of CAPM

- Risk Adjusted Discount Rate

$$\tilde{R}_j = \frac{\tilde{P}_e - P_0}{P_0} \quad (9)$$

Formal derivation

$$E[R_j] = R_f + [E[R_m] - R_f] \beta = \frac{E[\tilde{P}_e] - P_0}{P_0}$$

$$R_f + [E[R_m] - R_f] \beta = \frac{E[P_e]}{P_0} - 1 \quad (10)$$

$$1 + R_f + [E[R_m] - R_f] \beta = \frac{E[P_e]}{P_0}$$

- Certainty Equivalent Approach

$$1 + R_f + [E[R_m] - R_f] \beta = \frac{E[\tilde{P}_e]}{P_0}$$

$$1 + R_f = \frac{E[\tilde{P}_e] - [E[R_m] - R_f] \beta P_0}{P_0} \quad (11)$$

$$P_0 = \frac{E[\tilde{P}_0] - [E[R_m] - R_f] \beta P_0}{1 + R_f}$$

Arbitrage Pricing Theorem

- The concept is to show under what conditions a riskless, wealthless trade provides no expected rate of return.
 - The basic construct for this argument is the arbitrage portfolio w .
 - The arbitrage portfolio is defined as that set of purchases and sales that leaves wealth unchanged

$$w'1 = 0 \quad (12)$$

Factor Model of Returns

- The vector form of the factor model is

$$R_t = \bar{E} + bF_t + \epsilon_t \quad (13)$$

- where R is the vector of returns on assets,
- E is the vector of expected returns on assets,
- F is a matrix of factor loads relating the changes in the common factors with the fluctuations in asset returns, and
- ϵ is a vector of idiosyncratic risks.

Variance assumptions in the factor model

- $E[\epsilon_t \epsilon_t'] = \Theta$ a diagonal matrix (hence idiosyncratic risk).
- $E[\epsilon_t \epsilon_{t-s}'] = 0$ the errors are uncorrelated across time.
- $E[F_t F_t'] = \Psi$ the common factors are uncorrelated with the residuals.
- $E[F_t F_t'] = \Psi$ a diagonal matrix (by assumption of the estimation).

Arbitrage in the Factor Model

- Arbitrage and Expected Returns

$$w'R_t = w'\bar{E} + w'bF_t + w'\epsilon_t$$

$$\sum_i w_i R_{it} = \sum_i w_i \bar{E}_i + \sum_i w_i b_{i1} F_{1t} + \cdots \sum_i b_{ik} F_{kt} + \sum_i w_i \epsilon_{it} \quad (14)$$

- Variance in the arbitrage portfolio

$$\begin{aligned} E[w'RR'w] - [E[w'R]] [E[w'R]] &= \\ E([w'\bar{E} + w'bF + w'\epsilon] [\bar{E}'w + F'b'w + \epsilon'w]) - (w'\bar{E}\bar{E}'w) &= \\ E(w'\bar{E}\bar{E}'w + w'\bar{E}F'b'w + w'\bar{E}\epsilon'w + w'bF\bar{E}'w + w'bFF'b'w + \\ w'bF\epsilon'w + w'\epsilon\bar{E}'w + w'\epsilon F'b'w + w'\epsilon\epsilon'w) - (w'\bar{E}\bar{E}'w) &= \\ \Rightarrow V[w'R] = w'b\Psi b'w + w'\Theta w & \end{aligned} \quad (15)$$

- Therefore for riskless arbitrage $w'b = 0$.

$$\begin{aligned}\sum_i b_{i1} &= 0 \\ \sum_i b_{i2} &= 0 \\ &\vdots \\ \sum_i b_{ik} &= 0\end{aligned}\tag{16}$$

Arbitrage Efficiency

- Arbitrage efficiency then implies that
 - A no cost portfolio $w'1 = 0 \Rightarrow \sum_i w_i = 0$,
 - That yields no risk $w'b = 0$,
 - Implies no profit $w'\bar{E} = 0 \Rightarrow \sum_i w_i \bar{E}_i = 0$.

- The algebraic consequence of this statement is that

$$\bar{E}_i = \lambda_0 + \lambda_1 b_{i1} + \cdots \lambda_k b_{ik} \quad (17)$$

or the expected return on an asset is a linear function of the factor loadings. This result is identical to the CAPM results if there is a single factor.