

The Capital Asset Pricing Model: Lecture XXXIII

Charles B. Moss

October 31, 2014

- 1 Literature
- 2 Setting Up the Market
 - Perfect Markets
 - Firms
 - Consumers
 - Risk Equilibrium - Consumer
- 3 Role of the Riskless Asset
- 4 Supply of Stocks from Firms
- 5 Incorporating Risk Using CAPM

Literature

- Lintner, John. 1965. Security Prices, Risk, and Maximal Gain from Diversification. *Journal of Finance* 20, 587-615.
- Lintner, John. 1965. The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. *Review of Economics and Statistics* 47, 13-37.
- Mossin, Jan. 1966. Equilibrium in a Capital Asset Market. *Econometrica* 34, 768-783.
- Sharpe, W. F. 1964. Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. *Journal of Finance* 19, 425-442.

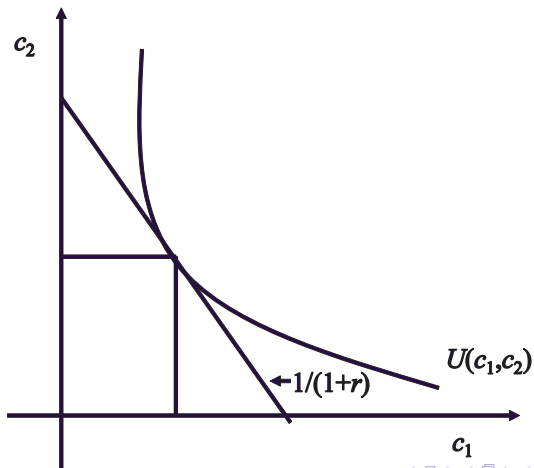
Setting Up the Market

- Perfect Markets: We assume that all markets are competitive for goods and investments.
 - All goods and investments are infinitely divisible.
 - Information is costless.
 - There are no transaction costs.
 - No individual is large enough to effect the price.

- Firms:
 - All goods are produced by firms.
 - These firms purchase the factors of production in the first period, produce output, and market their goods in the second period.
 - In addition, these firms do not have any capital of their own and must raise this capital by issuing stock.

- Consumers: Consumers begin with an endowment w_1 . The consumers choice problem (initially) is twofold.
 - First, the consumer must decide how much to consume in this period, c_1 , and how much to invest, h_1 . This investment will earn a rate of return $h_1(1 + R_f)$ which will be consumed in the second period, c_2 .
 - Second, the consumer must decide how to invest h_1 , that is how to divide it up between a wide array of assets.
 - In general, this implies two decision dimensions.
 - The first is intertemporal (across time). In this decision the consumer has a time preference, the preference between consuming now and consuming later (Figure 1)

Figure : Intertemporal Consumption



- Continued
 - Continued
 - The second is the risk or uncertainty on the investment. Both of these questions can be represented in the utility function.
 - Market Equilibrium: Assuming that firms supply investment and consumers demand investment opportunities, we hypothesize that there exists an equilibrium where the supply of stocks equals the demand of stocks.

- Continued

- Risk Equilibrium from the Consumer's Point of View

- At the outset, we assume that consumers are risk averse and that risk can be characterized using the normal distribution function.
 - Given these assumptions, we can use the Expected Value-Variance, or in this case the Expected Value-Standard Deviation approach to expected utility/risk efficiency.
 - The mean of the portfolio is written as

$$\mu_p = \sum_{i=1}^N \mu_i \tilde{x}_i \quad (1)$$

where μ_p is the expected return from the portfolio, μ_i is the expected return on a specific asset, and x_i is the level of asset i held in a specific portfolio.

- Continued
 - Continued
 - Similarly, the standard deviation of the portfolio can be written as

$$\sigma_p = \left[\sum_{j=1}^N \sum_{i=1}^N \sigma_{ij} \tilde{x}_i \tilde{x}_j \right]^{\frac{1}{2}} \quad (2)$$

where σ_p is the standard deviation of a particular portfolio and σ_{ij} is the covariance between asset i and asset j .

- In addition, if we impose the portfolio balance condition

$$\sum_{i=1}^N \tilde{x}_i = 1 \quad (3)$$

- Continued
 - Continued
 - We can reformulate the risk measure, standard deviation of the portfolio, to analyze the contribution of each asset to the overall risk of the portfolio

$$\begin{aligned}\sigma_p &= \frac{\sigma_p^2}{\sigma_p} \\ &= \frac{\sum_{j=1}^N \sum_{i=1}^N \sigma_{ij} \tilde{X}_i \tilde{X}_j}{\sigma_p} \\ &= \sum_{j=1}^N \tilde{X}_j \left(\frac{\sum_{i=1}^N \sigma_{ij} \tilde{X}_i}{\sigma_p} \right)\end{aligned}\tag{4}$$

- Continued

- The risk of a particular asset x_j is then dependent on weighted covariances between asset j and the returns in the rest of the portfolio.
- Remember that the x_i s are weights in the general portfolio. This raises two points:
 - First, note that the risk of an individual asset depends on the portfolio weights and the risk of the portfolio.
 - Second, the risk of a particular asset depends both on its own variance and the variance of the remaining assets in the portfolio. Thus, as the number of assets becomes large and the portfolio becomes well diversified, the risk of a particular asset is more dependent on the covariance with other assets in the portfolio than on its own risk.

- Continued

- Following our previous discussions of Expected Value-Variance frontier, we assume that consumers choose the portfolio that minimizes risk for any given level of expected income. However, in this case risk is parameterized by the standard deviation instead of the variance.

$$\begin{aligned} \min \sigma_p &= \left(\sum_{j=1}^N \sum_{i=1}^N \sigma_{ij} \tilde{x}_j \tilde{x}_i \right)^{\frac{1}{2}} \\ \text{s.t. } \mu_p &= \sum_{i=1}^N \mu_i \tilde{x}_i = \mu^* \\ \sum_{i=1}^N \tilde{x}_i &= 1 \end{aligned} \tag{5}$$

- Continued
 - Writing out the Lagrangian

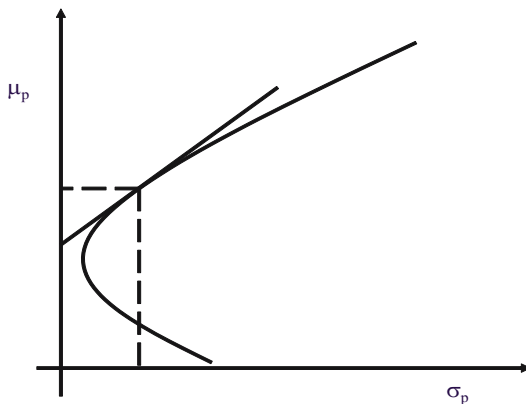
$$\begin{aligned} L &= \sigma_p + \lambda_1 \left(\mu_p - \sum_{i=1}^N \mu_i \tilde{x}_i \right) + \lambda_2 \left(1 - \sum_{i=1}^N \tilde{x}_i \right) \\ \frac{\partial L}{\partial \tilde{x}_k} &= \frac{\partial \sigma_p}{\partial \tilde{x}_k} - \lambda_1 \mu_k - \lambda_2 = 0 \\ \frac{\partial L}{\partial \lambda_1} &= \mu_p - \sum_{i=1}^N \mu_i \tilde{x}_i = 0 \\ \frac{\partial L}{\partial \lambda_2} &= 1 - \sum_{i=1}^N \tilde{x}_i = 0 \end{aligned} \tag{6}$$

- Continued
 - Comparing any two investments k and l yields

$$\begin{aligned}\frac{\partial \sigma_p}{\partial \tilde{x}_k} - \lambda_1 \mu_k &= \frac{\partial \sigma_p}{\partial \tilde{x}_l} - \lambda_1 \mu_l \\ \mu_k - \mu_l &= \frac{1}{\lambda_1} \left(\frac{\partial \sigma_p}{\partial \tilde{x}_l} - \frac{\partial \sigma_p}{\partial \tilde{x}_k} \right)\end{aligned}\tag{7}$$

This relationship is depicted graphically in Figure 2.

Figure : Portfolio Tangency



- Continued

- Next, we sum the relationship in Equation 7 across the portfolio to yield

$$\begin{aligned} \sum_{i=1}^N \tilde{x}_i (\mu_j - \mu_i) &= \frac{1}{\lambda_1} \left(\sum_{i=1}^N \tilde{x}_i \frac{\partial \sigma_p}{\partial \tilde{x}_j} - \sum_{i=1}^N \tilde{x}_i \frac{\partial \sigma_p}{\partial \tilde{x}_i} \right) \\ \mu_j - \mu_p &= \frac{1}{\lambda_1} \left(\frac{\partial \sigma_p}{\partial \tilde{x}_j} - \sum_{i=1}^N \tilde{x}_i \frac{\partial \sigma_p}{\partial \tilde{x}_i} \right) \\ \sigma_p &= \sum_{j=1}^N \tilde{x}_j \left(\frac{\sum_{i=1}^N \sigma_{ij} \tilde{x}_i}{\sigma_p} \right) \Rightarrow \frac{\partial \sigma_p}{\partial \tilde{x}_k} = \frac{\sum_{i=1}^N \sigma_{ik} \tilde{x}_i}{\sigma_p} \end{aligned} \quad (8)$$

- Continued

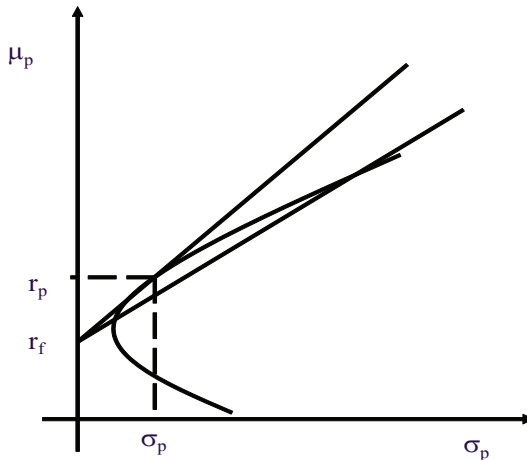
- Thus, summing the derivative $\sum_{i=1}^N \tilde{x}_i \partial \sigma_p / \partial \tilde{x}_i$ term implies

$$\begin{aligned} \sum_{i=1}^N \tilde{x}_i \frac{\partial \sigma_p}{\partial \tilde{x}_i} &= \sum_{i=1}^N \tilde{x}_i \sum_{j=1}^N \frac{\sigma_{ij} \tilde{x}_j}{\sigma_p} = \sum_{j=1}^N \sum_{i=1}^N \frac{\sigma_{ij} \tilde{x}_i \tilde{x}_j}{\sigma_p} = \sigma_p \\ \mu_j - \mu_p &= \frac{1}{\lambda_1} \left(\sum_{i=1}^N \frac{\sigma_{ij} \tilde{x}_i \tilde{x}_j}{\sigma_p} - \sigma_p \right) \\ \mu_j - \mu_p &= \frac{1}{\lambda_1} \left(\frac{\sigma_{jp}}{\sigma_p} - \sigma_p \right) \end{aligned} \quad (9)$$

The Role of the Riskless Asset

- The equilibrium presented above does not yield an estimable representation because different investors may have different risk preferences. One way around this ambiguity is to introduce a riskless asset.
 - The riskless asset reduces the potential number of efficient portfolios to a single portfolio (as depicted in Figure 3).

Figure : Equilibrium with Risk Free Asset



- Continued

- Within this equilibrium, there is only one efficient portfolio of assets. Any degree of risk aversion can construct a risk preferred position by holding a combination of the single efficient portfolio and borrowing or lending at the riskless rate.
- Substituting x_m for x_p or letting the index portfolio be the market efficient portfolio, we have

$$\mu_j - \mu_m = \frac{1}{\lambda_1} \left(\frac{\partial \sigma_m}{\partial \tilde{x}_j} - \sigma_m \right) \quad (10)$$

- Continued

- Consider the same relationship, only let μ_j be the risk-free asset (thus $\partial\sigma_m/\partial x_f = 0$ and $\mu_f = r_f$)

$$\frac{1}{\lambda_1} = \frac{(\mu_m - r_f)}{\sigma_m} \quad (11)$$

Thus

$$\begin{aligned} \mu_j &= r_f + \left[\frac{\mu_m - r_f}{\sigma_m} \right] \frac{\partial\sigma_m}{\partial \tilde{x}_j} \\ &= r_f + \left[\frac{\mu_m - r_f}{\sigma_m} \right] \frac{\sigma_{jm}}{\sigma_m} \\ &= r_f \beta_j (\mu_m - r_f) \\ \beta_j &= \frac{\sigma_{jm}}{\sigma_m^2} \\ r_{it} &= \alpha + \beta_i r_{mt} + \epsilon_{it} \end{aligned} \quad (12)$$

Supply of Stocks from the Firm

- We start by assuming that each firm sells stocks at a price P_i .
- Investors are willing to bid on these stocks based on the future value of the firm at the end of the year, V_i .
- The bid price and the value at the end of the year determine the rate of return

$$R_i = \frac{V_i - P_i}{P_i} \quad (13)$$

Given that the future value of the firm implies some risk, the rate of return is risky. In addition, given the preceding proof we know that investors value the investment under the capital market equilibrium.

- Mathematically, the price and value of the market portfolio becomes

$$P_m = \sum_{i=1}^N P_i \text{ and } V_m = \sum_{i=1}^N V_i \quad (14)$$

Thus for the market portfolio

$$V_m = P_m (1 + R_M) \quad (15)$$

and then for any stock j

$$\frac{E(V_j) - P_j}{P_j} = R_f + \left[\frac{E(R_m) - R_f}{\sigma_m} \right] \frac{\sigma_{jm}}{P_j \sigma_m} \quad (16)$$

Incorporating Risk Using CAPM

- Both risk adjustments come directly from Equation 16.
 - The first approach is referred to as the risk adjusted discount rate (RADR). Reducing the preceding expression to the CAPM formula

$$R_j = R_f + \beta_j (R_m - R_f) \quad (17)$$

This risk adjusted discount rate can be used in present value analysis

$$PV_t = \sum_{i=1}^N \frac{E[CF_{t+i}|\Omega_t]}{(1 + [R_f + \beta_j \{ R_m - R_f \}])^i} \quad (18)$$

- Continued
 - Reformulating Equation 17 slightly

$$\frac{1}{\beta_j} (R_j - R_f) = R_M - R_f \quad (19)$$

This approach leads to transforming the annual rates of return into "certainty equivalents" based on the market portfolio

$$PV_t = \sum_{i=1}^N \frac{\frac{1}{\beta_j} E[CF_{t+i} | \Omega_t]}{(1 + R_m)^i} \quad (20)$$