

Generalized Stochastic Dominance with Respect to a Function: Lecture XXX

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October 31, 2014

- 1 Implications of Risk Aversion
- 2 Stochastic Dominance with Respect to a Function

Implications of Risk Aversion

- Next, we use the result from Pratt to show that this inequality holds for all utility functions more risk averse than $r(x)$. A brief review of Pratt.
 - This article titled "Risk Aversion in the Small and Large" appeared in *Econometrica* 32, 1964, 122-136 and derives the definition of what Pratt refers to as **local risk aversion**

$$r(x) = -\frac{u''(x)}{u'(x)} \quad (1)$$

- Continued

- This definition is based off the definition of the risk premium π . In this article the risk premium is defined as a function of initial wealth x . Thus, as we have come to know and love the risk premium is defined as that certain amount $\pi(x, z)$ where z is a random event such that

$$u(x + E(z) - \pi(x, z)) = E\{u(x + z)\} \quad (2)$$

- Letting $E(z)$ go to zero (or letting the random investment be actuarially neutral and taking the second order Taylor series expansion of both sides yields

$$\begin{aligned} u(x) - \pi u'(x) + o(\pi^3) &= E\left\{u(x) + zu'(z) + \frac{1}{2}u''(x) + o(z^3)\right\} \\ u(x) - \pi u'(x) + o(\pi^3) &= u(x) + \frac{1}{2}\sigma_z^2 u''(x) + o(z^3) \end{aligned} \quad (3)$$

- Continued

- Assuming that the two second order terms go to zero for a small gamble and rearranging yields

$$\begin{aligned}\pi(x, z) &= -\frac{1}{2} \frac{u''(x)}{u'(x)} \sigma_z^2 \\ &= \frac{1}{2} r(x) \sigma_z^2 \text{ given } r(x) = -\frac{u''(x)}{u'(x)}\end{aligned}\tag{4}$$

- Theorem 1.** Which is used by Meyer, then states that

$$r_1(x) \geq r_2(x) \Rightarrow \left\{ \begin{array}{l} \pi_1(x, z) \geq \pi_2(x, z) \\ u_1(u_2^{-1}(t)) \text{ is strictly convex at } t \\ \frac{u_1(y) - u_1(x)}{u_1(w) - u_1(v)} \leq \frac{u_2(y) - u_2(x)}{u_2(w) - u_2(v)} \text{ for all } v, w, x, y \text{ with } v < w \leq x < y \end{array} \right.\tag{5}$$

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- The point of the Pratt theorem is that for any individual more risk averse than $r(x)$ the inequality will also hold. Thus, $F(x)$ will be preferred by all agents who are more risk averse than the index individual.
- Theorem 2.** Consider cumulative distribution functions $F(x)$ and $G(x)$, such that their probability functions cross a finite number of times, then there is an increasing continuous twice differentiable function $r(x)$, such that

$$\int_0^y [G(x) - F(x)] dr(x) \geq 0, \forall y \in [0, 1] \quad (6)$$

if and only if $G(x)$ is at least as risky as $F(x)$ by Definition 2.

- Next Meyer shows that his proposed definition of increasing risk yields a partial ordering over the set of cumulative distributions. In order for the definition to provide a partial ordering, it is necessary to show that the definition is
 - Binary,
 - Transitive,
 - Reflexive, and
 - Antisymmetric

- Defining the relationship that G is at least as risky as F as $F >_r G$. To show transitivity then requires $F >_r G$ and $G >_r H$ implies that $F >_r H$.
 - This proof is completed by defining $U(r(x))$ as the set of all utility functions such that $u(x)$ is more risk averse than $r(x)$. Given this definition, $F >_r G$ by all individuals such that $u(x) \in U(r_1(x))$ and $G >_r H$ by all individuals such that $u(x) \in U(r_2(x))$.
 - Given these two definitions, we want to derive $r_3(x)$ as the level of risk aversion such that $U(r_3(x)) = U(r_1(x)) \cap U(r_2(x))$

$$-\frac{u_3''(x)}{u_3'(x)} = \max \left(-\frac{u_1''(x)}{u_1'(x)}, -\frac{u_2''(x)}{u_2'(x)} \right) \quad (7)$$

- Given that expected utility is transitive for economic agents, this result is sufficient to show that individuals who prefer G to H and F to G also prefer F to H .

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- Theorem 3.** $F <_r G$ and $G <_r F$ if and only if $F = G$.

$$\begin{aligned} \int_0^y [G(x) - F(x)] dr_1(x) &\geq 0 \quad \forall y \in [0, 1] \\ \int_0^y [F(x) - G(x)] dr_2(x) &\geq 0 \quad \forall y \in [0, 1] \end{aligned} \quad (8)$$

$$-\frac{u_3''(x)}{u_3'(x)} = \max \left(-\frac{u_1''(x)}{u_1'(x)}, -\frac{u_2''(x)}{u_2'(x)} \right)$$

$$\begin{aligned} \int_0^y [G(x) - F(x)] dr_3(x) &\geq 0 \quad \forall y \in [0, 1] \quad \text{and} \\ \int_0^y [G(x) - F(x)] dr_3(x) &\geq 0 \quad \forall y \in [0, 1] \end{aligned} \quad (9)$$

- Continued

- Theorem 4.** For any two cumulative distributions $F(x)$ and $G(x)$ such that their probability functions cross a finite number of times there exists an increasing continuous twice differentiable function, $r(x)$, such that

$$\int_0^y [G(x) - F(x)] dr(x) \geq 0 \quad \forall y \in [0, 1] \quad (10)$$

Stochastic Dominance with Respect to a Function

- Meyer, Jack. 1977. Choice among Distributions. *Journal of Economic Theory* 14, 326-336.
 - The general idea of the manuscript is to restrict the risk aversion coefficient for stochastic dominance to those risk aversion coefficients in a given interval $r_1(x) < r(x) < r_2(x)$.
 - This problem will be solved by finding the utility function $u(x)$ which satisfies

$$r_1(x) \leq -\frac{u''(x)}{u'(x)} \leq r_2(x) \quad \forall x \in [0, 1] \quad (11)$$

and minimizes

$$\int_0^1 [G(x) - F(x)] u'(x) dx \quad (12)$$

- Continued

- Given that this integral yields the expected value of $F(x)$ minus the expected value of $G(x)$, the minimum will be greater than zero if $F(x)$ is preferred to $G(x)$ by all agents who prefer $F(x)$ to $G(x)$.
- If the minimum is less than zero, then the preference is not unanimous for all agents whose risk aversion coefficients are in the state range.
- Another problem is that utility is invariant to a linear transposition. Thus, we must stipulate that $u'(0) = 1$.

- The problem is then to use the control variable $-u''(x)/u'(x)$ to maximize the objective function

$$-\int_0^1 [G(x) - F(x)] u'(x) dx \quad (13)$$

subject to the equation of motion

$$(u'(x))' = -u'(x) \frac{u''(x)}{u'(x)} \quad (14)$$

- And the control constraints

$$\begin{aligned} -\frac{u''(x)}{u'(x)} - r_1(x) &\geq 0 \\ \frac{u''(x)}{u'(x)} + r_2(x) &\geq 0 \end{aligned} \tag{15}$$

with the initial condition $u'(x) = 1$.

- Rewriting the problem, substituting $z(x) = -u''(x)/u'(x)$ yields

$$\begin{aligned} \max & - \int_0^1 [G(x) - F(x)] u'(x) dx \\ \text{s.t. } & (u'(x))' = u'(x) z(x) \\ & -z(x) + r_1(x) \leq 0 \\ & z(x) - r_2(x) \leq 0 \end{aligned} \tag{16}$$

- The traditional Hamiltonian for this problem then becomes

$$H = -[G(x) - F(x)] u'(x) + \lambda(x) (u'(x) z(x)) \quad (17)$$

Following the basic Pontryagin results, a given path for control variable is optimum given three conditions

- Optimality Condition

$$\frac{\partial H}{\partial z} = 0 \quad (18)$$

- Continued
 - Multiplier or Costate condition

$$-\frac{\partial H}{\partial u'} = \lambda'(x) \quad (19)$$

- Equation of Motion

$$\frac{\partial H}{\partial \lambda} = (u'(x))' \quad (20)$$

- However, in the current scenario the Hamiltonian has to be amended to account for the two inequality constraints. Specifically,

$$L = H + \mu_1(x)(-z(x) + r_1(x)) + \mu_2(x)(z(x) - r_2(x)) \quad (21)$$

- First examining the derivatives of the Lagrangian with respect to $z(x)$, the control variable. We see that

$$\frac{\partial L}{\partial z(x)} = \lambda(x) u'(x) - \mu_1(x) + \mu_2(x) = 0 \quad (22)$$

- Given the restriction that the Lagrange multipliers must be nonnegative at optimum, we can see that

$$z(x) = \begin{cases} r_1(x) & \text{if } \lambda(x) u'(x) \geq 0 \Rightarrow \mu_1(x) \geq 0 \text{ with } \mu_2(x) = 0 \\ r_2(x) & \text{if } \lambda(x) u'(x) \leq 0 \Rightarrow \mu_2(x) \geq 0 \text{ with } \mu_1(x) = 0 \end{cases} \quad (23)$$

Thus, the minimum value of the integral occurs at one of the boundaries. The question is then: What determines which boundary?

- To examine this we begin with the Costate condition

$$-\frac{\partial L}{\partial u'(x)} = [G(x) - F(x)] - \lambda(x) z(x) = \lambda'(x) \quad (24)$$

Given this expression, we can work backward from the transversality condition. Specifically, the transversality condition for this control problem specifies that

$$\lambda(1) u'(1) = 0. \quad (25)$$

- Given this boundary condition, the value of $\lambda(x) u'(x)$ can be defined by the integral

$$\begin{aligned}\int_x^1 \frac{d(\lambda(s) u'(s))}{ds} ds &= \lambda(1) u'(1) - \lambda(x) u'(x) \\ &= -\lambda(x) u'(x) \\ \lambda(x) u'(x) &= -\int_x^1 \frac{d(\lambda(s) u'(s))}{ds} ds\end{aligned}\tag{26}$$

- Hence to complete the derivation, we must determine the value of the derivative under the integral

$$\frac{d(\lambda(x) u'(x))}{dx} = \lambda'(x) u'(x) + \lambda(x) u''(x) \quad (27)$$

Substituting for $\lambda'(x)$ from the Costate condition yields

$$\begin{aligned} \frac{d(\lambda(x) u'(x))}{dx} = & ([G(x) - F(x)] - \lambda(x) z(x)) u'(x) \\ & + \lambda(x) u''(x) \end{aligned} \quad (28)$$

- Substituting for $z(x)$ and canceling like terms yields

$$\frac{d(\lambda(x) u'(x))}{dx} = [G(x) - F(x)] u'(x) \quad (29)$$

Yielding the expression

$$\lambda(x) u'(x) = - \int_x^1 [G(x) - F(x)] u'(x) dx \quad (30)$$

- Merging this result with the previous derivations, we have the infamous Theorem 5: An optimal control $-u''(x)/u'(x)$ which maximizes

$$\begin{aligned}
 & - \int_0^1 [G(x) - F(x)] u'(x) dx \\
 \text{s.t. } & r_1(x) \leq -\frac{u''(x)}{u'(x)} \leq r_2(x) \\
 & u'(x) = 1
 \end{aligned} \tag{31}$$

is given by

$$-\frac{u''(x)}{u'(x)} = \begin{cases} r_1(x) & \text{if } \int_x^1 [G(x) - F(x)] u'(x) dx \leq 0 \\ r_2(x) & \text{if } \int_x^1 [G(x) - F(x)] u'(x) dx \geq 0 \end{cases} \tag{32}$$

- The empirical idea is then to determine whether one distribution is dominated by another distribution in the second degree with respect to a particular set of preferences. To do this we want to know if the integral switches from negative to positive within the range of risk aversion coefficients.
- Theorem 5 states that the risk aversion which maximizes the integral will be found at one boundary or the other depending on the sign of the integral.