

Implications of Increasing Spreads: Lecture XXIX

Charles B. Moss

November 3, 2014

- 1 Increasing Spread
- 2 Implications of Risk Aversion

Increasing Spread

- Theorem 1: Consider cumulative distribution functions $F(x)$ and $G(x)$, then there is an increasing continuous function $r(x)$ such that

$$\int_0^y [G(x) - F(x)] dr(x) \geq 0 \forall y \in [0, 1] \quad (1)$$

if and only if $G(x)$ is at least as risky as $F(x)$ by Definition 1.

- Proof: Assume there exists an increasing twice differentiable $r(x)$ such that

$$\int_0^y [G(x) - F(x)] dr(x) \geq 0, \forall y \in [0, 1]. \quad (2)$$

- $z = r(x)$ and define functions $G^*(.)$ where $G^*(z) = G(r^{-1}(x))$ and $F^*(.)$ where $F^*(r^{-1}(x))$. Then,

$$\int_0^y [G^*(r(x)) - F^*(r(x))] dr(x) \geq 0, \forall y \in [0, 1] \text{ or} \quad (3)$$

$$\int_{r(0)}^{r(y)} [G^*(z) - F^*(z)] dz \geq 0, \forall y \in [0, 1]$$

- Rewriting slightly

$$\int_{r(0)}^{r(y)} [G^*(z) - F^*(z)] u'(z) dz \geq 0 \quad (4)$$

$$\Rightarrow \int_{r(0)}^{r(y)} [G^*(z) - F^*(z)] du(z) \geq 0.$$

- A couple of notes on this point
 - If you made the assumption that $r(x)$ is a one-to-one mapping, we have simply changed the definition of the cumulative distribution defining it on the variable z which is related to the original variable x by $z = r(x)$.
 - This assumption would be implied by the imposition $x = r^{-1}(z)$.
 - Along the same lines, we really haven't changed the bounds of integration.
 - We have only mapped them into the variable z .
 - Mapping the transformation back to x by $v(x) = u(r(x))$, the integral becomes

$$\int_0^1 [G(x) - F(x)] dv(x) \geq 0$$

$$\int_0^1 v(x) dF(x) \geq \int_0^1 v(x) dG(x) \quad (5)$$

Implications of Risk Aversion

- Next, we use the result from Pratt to show that this inequality holds for all utility functions more risk averse than $r(x)$. A brief review of Pratt.
 - This article titled "Risk Aversion in the Small and Large" appeared in *Econometrica* 32, 1964, 122-136 and derives the definition of what Pratt refers to as **local risk aversion**

$$r(x) = -\frac{u''(x)}{u'(x)} \quad (6)$$

This definition is based off the definition of the risk premium π . In this article the risk premium is defined as a function of initial wealth x . Thus, as we have come to know and love the risk premium is defined as that certain amount $\pi(x, z)$ where z is a random event such that

$$u(x + E(z) - \pi(x, z)) = E\{u(x + z)\} \quad (7)$$

- Letting $E(z)$ go to zero (or letting the random investment be actuarially neutral and taking the second order Taylor series expansion of both sides yields

$$u(x) - \pi u'(x) + o(\pi^3) = E \left\{ u(x) + zu'(z) + \frac{1}{2} u''(x) + o(z^3) \right\}$$

$$u(x) - \pi u'(x) + o(\pi^3) = u(x) + \frac{1}{2} \sigma_z^2 u''(x) + o(z^3) \quad (8)$$

- Assuming that the two second order terms go to zero for a small gamble and rearranging yields

$$\pi(x, z) = -\frac{1}{2} \frac{u''(x)}{u'(x)} \sigma_z^2$$

$$= \frac{1}{2} r(x) \sigma_z^2 \text{ given } r(x) = -\frac{u''(x)}{u'(x)} \quad (9)$$

- **Theorem 1.** Which is used by Meyer, then states that

$$\Rightarrow \left\{ \begin{array}{l} r_1(x) \geq r_2(x) \\ \pi_1(x, z) \geq \pi_2(x, z) \\ u_1(u_2^{-1}(t)) \text{ is strictly convex at } t \\ \frac{u_1(y) - u_1(x)}{u_1(w) - u_1(v)} \leq \frac{u_2(y) - u_2(x)}{u_2(w) - u_2(v)} \text{ for all } v, w, x, y \text{ with } v < w \leq x < y \end{array} \right. \quad (10)$$

The point of the Pratt theorem is that for any individual more risk averse than $r(x)$ the inequality will also hold. Thus, $F(x)$ will be preferred by all agents who are more risk averse than the index individual.

- **Theorem 2.** Consider cumulative distribution functions $F(x)$ and $G(x)$, such that their probability functions cross a finite number of times, then there is an increasing continuous twice differentiable function $r(x)$, such that

$$\int_0^y [G(x) - F(x)] dr(x) \geq 0, \forall y \in [0, 1] \quad (11)$$

if and only if $G(x)$ is at least as risky as $F(x)$ by Definition 2.

- Next Meyer shows that his proposed definition of increasing risk yields a partial ordering over the set of cumulative distributions. In order for the definition to provide a partial ordering, it is necessary to show that the definition is
 - Binary,
 - Transitive,
 - Reflexive, and
 - Antisymmetric
- Defining the relationship that G is at least as risky as F as $F >_r G$. To show transitivity then requires $F >_r G$ and $G >_r H$ implies that $F >_r H$.

- This proof is completed by defining $U(r(x))$ as the set of all utility functions such that $u(x)$ is more risk averse than $r(x)$. Given this definition, $F >_r G$ by all individuals such that $u(x) \in U(r_1(x))$ and $G >_r H$ by all individuals such that $u(x) \in U(r_2(x))$.
- Given these two definitions, we want to derive $r_3(x)$ as the level of risk aversion such that

$$U(r_3(x)) = U(r_1(x)) \cap U(r_2(x))$$

$$-\frac{u_3''(x)}{u_3'(x)} = \max \left(-\frac{u_1''(x)}{u_1'(x)}, -\frac{u_2''(x)}{u_2'(x)} \right) \quad (12)$$

- Given that expected utility is transitive for economic agents, this result is sufficient to show that individuals who prefer G to H and F to G also prefer F to H .

- **Theorem 3.** $F <_r G$ and $G <_r F$ if and only if $F = G$.

$$\begin{aligned} \int_0^y [G(x) - F(x)] dr_1(x) &\geq 0 \quad \forall y \in [0, 1] \\ \int_0^y [F(x) - G(x)] dr_2(x) &\geq 0 \quad \forall y \in [0, 1] \end{aligned} \quad (13)$$

$$-\frac{u_3''(x)}{u_3'(x)} = \max \left(-\frac{u_1''(x)}{u_1'(x)}, -\frac{u_2''(x)}{u_2'(x)} \right)$$

$$\begin{aligned} \int_0^y [G(x) - F(x)] dr_3(x) &\geq 0 \quad \forall y \in [0, 1] \quad \text{and} \\ \int_0^y [G(x) - F(x)] dr_3(x) &\geq 0 \quad \forall y \in [0, 1] \end{aligned} \quad (14)$$

- **Theorem 4.** For any two cumulative distributions $F(x)$ and $G(x)$ such that their probability functions cross a finite number of times there exists an increasing continuous twice differentiable function, $r(x)$, such that

$$\int_0^y [G(x) - F(x)] dr(x) \geq 0 \quad \forall y \in [0, 1] \quad (15)$$