

The Farm Portfolio Problem: Lecture XXIII

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An Empirical Model of Mean-Variance

- Deriving the EV frontier
 - Let us begin with the traditional portfolio model. Assume that we want to minimize the variance associated with attaining a given level of income. To specify this problem we assume a variance matrix

$$\begin{aligned} \max_x f(x) &= x' \Omega x \\ \text{s.t. } Ax &= b \end{aligned}$$
$$\Omega = \begin{pmatrix} 924.41 & 458.52 & 202.22 & 135.22 \\ 458.52 & 761.29 & 452.99 & 72.55 \\ 202.22 & 452.99 & 490.11 & 109.09 \\ 135.22 & 72.25 & 109.09 & 284.17 \end{pmatrix} \quad (1)$$

- Continued
 - Consider the following constraints

$$Ax = b$$

$$\begin{pmatrix} 8.119 & 11.366 & 6.298 & 8.014 \\ 1.0 & 1.0 & 1.0 & 1.0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7.0 \\ 1.0 \end{pmatrix}$$

$$8.119x_1 + 11.366x_2 + 6.298x_3 + 8.014x_4 = 7.000$$

$$x_1 + x_2 + x_3 + x_4 = 1.000 \quad (2)$$

where the first constraint is the income constraint and the second constraint is the portfolio balance constraint.

- In this initial formulation we find that the optimum solution is x which yields a variance of 228.25

$$x = \begin{pmatrix} 0.1161 \\ -0.10666 \\ 0.39508 \end{pmatrix} \quad (3)$$

Computer Code for Portfolio Analysis

- Parts of GAMS Programs
 - Sets
 - Tables
 - Parameters
 - Variables
 - Equations
 - Model Setup
- Starting with the basic model of portfolio choice

$$\begin{aligned} \min_x & x' \Omega x \\ \text{s.t.} & x' \mu \geq Y^* \end{aligned} \quad (4)$$

- Freund showed that the expected utility of a normally distributed gamble given negative exponential preferences could be written as

$$E[U(x)] = \mu(x) - \frac{\rho}{2} \sigma^2(x)$$

Freund's Florida Farm Example

- We start with Freund's formulation.
 - Freund has four sets of constraints
 - Land constraints
 - Production capital constraints
 - Managerial Labor constraints
 - Activity level to generate \$ 100.00 in profit
 - These constraints are presented in Table 1.

Table: Freund's Constraints

Scarce Resource	Early Irish Potatoes	Corn	Beef	Fall Cabbage	Resource Constraint
<i>Land Constraints</i>					
Jan-June	1.199	1.382	2.776	0.000	60.0
July-Dec	0.000	1.382	2.776	0.482	60.0
<i>Production Capital</i>					
Period 1	1.064	0.484	0.038	0.000	24.0
Period 2	-2.064	0.020	0.107	0.229	12.0
Period 3	-2.064	-1.504	-1.145	-1.229	0.0
<i>Managerial Labor</i>					
Period 1	5.276	4.836	0.000	0.000	799.0
Period 2	2.158	4.561	0.000	4.198	867.0
Period 3	0.000	4.561	0.000	13.606	783.0

Unit Level \$100.00

- Continued
 - The Variance matrix for the problem is

$$\Omega = \begin{pmatrix} 7304.69 & 903.89 & -688.73 & -182.05 \\ & 620.16 & -471.14 & 110.43 \\ & & 1124.64 & 750.69 \\ & & & 3689.53 \end{pmatrix} \quad (6)$$

- Continued
 - The maximization problem can then be written as

$$\max_x \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}' \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} - \frac{\rho}{2} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' \Omega \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\begin{pmatrix} 1.199 & 1.382 & 2.776 & 0 \\ 0 & 1.382 & 2.776 & 0.482 \\ 1.064 & 0.484 & 0.038 & 0 \\ -2.064 & 0.020 & 0.107 & 0.229 \\ -2.046 & -1.504 & -1.145 & -1.229 \\ 5.276 & 4.836 & 0 & 0 \\ 2.158 & 4.561 & 0 & 4.198 \\ 0 & 4.146 & 0 & 13.607 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \leq \begin{pmatrix} 60.0 \\ 60.0 \\ 24.0 \\ 12.0 \\ 0.0 \\ 799.0 \\ 867.0 \\ 787.0 \end{pmatrix} \quad (7)$$

- Continued

- Using $\rho = 1/1250.0$ we obtain an optimal solution under risk of

$$x = \begin{pmatrix} 10.29 \\ 26.76 \\ 2.68 \\ 32.35 \end{pmatrix} \quad (8)$$

- The objective function for this optimum solution is 5,383.08. Putting r equal to zero yields an objective function of 9,131.11 with a allocation of

$$x = \begin{pmatrix} 22.14 \\ 0.00 \\ 11.62 \\ 57.55 \end{pmatrix} \quad (9)$$

- Question: How does the current solution compare to the risk averse solution? Which crop makes the greatest gain? Which crop has the largest loss? Why?

Quantifying Gains to Diversification

- Quantifying Gains to Risk Diversification Using Certainty Equivalence in a Mean-Variance Model: An Application to Florida Citrus.
 - The traditional formulation of the mean-variance rules begins with the negative exponential utility function

$$U[W(x)] = -\exp(-\rho W[x]) \quad (10)$$

- Our previous discussion indicated that this expected utility can be rewritten under normality as

$$E[U(W\{x\})] = -\exp\left[-\rho\left(\mu(x) - \frac{\rho}{2}\sigma^2(x)\right)\right] \quad (11)$$

- Hence, our tradition of maximizing

$$z = \mu(x) - \frac{\rho}{2}\sigma^2(x) \quad (12)$$

- Continued

- The implications of this objective function is actually much broader, however. Solving the negative exponential utility function for wealth

$$\begin{aligned}
 U[W^*(x)] &= -e^{-\rho W^*(x)} \\
 \ln[-U[W^*(x)]] &= -\rho W^*(x) \\
 W^*(x) &= -\frac{1}{\rho} \ln[-U[W^*(x)]] \\
 W^*(x) &= \mu(x) - \frac{\rho}{2} \sigma^2(x)
 \end{aligned} \tag{13}$$

- Other implications include the interpretation of the shadow values of the constraint as changes in certainty equivalence.
 - For example, given the original specification of the objective function, the shadow values of the second land constraint is 34.73 and the shadow value of the first capital constraint is 93.98.
 - These values are then the price of each input under uncertainty

Multiperiod Portfolio

- Moss, Charles B., Allen M. Featherstone, and Timothy G. Baker. "Agricultural Assets in an Efficient Multiperiod Investment Portfolio." *Agricultural Finance Review* 49(1987): 82-94.
- Historically, ownership of agricultural assets has been dominated by farmer equity and debt capital.
 - The implication of this form of ownership are increased variability in the return on equity to farmers.
 - A direct manifestation of the unwillingness of nonfarm investors to invest in agriculture can be seen in the unexplained premium on farm assets in the Capital Asset Pricing Model.
- This study examines whether autocorrelation in the returns on farm assets versus other assets may explain the discrepancy.
- Autocorrelation in farm returns refers to the tendency of increased returns to persist over time. Mathematically

- Given this vector of returns, the problem is to design the expected value/variance problem for holding a given portfolio of assets over several periods. Mathematically, this produces two problems:
 - Given the autoregressive structure of the problem, what is the expected return?

$$\begin{aligned}(I - AL) \epsilon_t &= \alpha_0 \\ \epsilon_t &= (I - AL)^{-1} \alpha_0\end{aligned}\tag{15}$$

- A similar problem involves the variance matrix. Using the autoregressive estimation above, the variance matrix for the investment can be written as