

Derivation of the Expected Value-Variance Frontier with a Risk-Free Asset: Lecture XXII

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- 1 Introduction of a Risk-free Asset
 - Mechanics
- 2 Capital Market Line

Introduction of a Risk-free Asset

- If a risk-free asset is introduced into the portfolio, the efficient set of portfolios becomes a straight line between the risk-free asset and a tangency on the ES frontier as depicted in Figure 1.

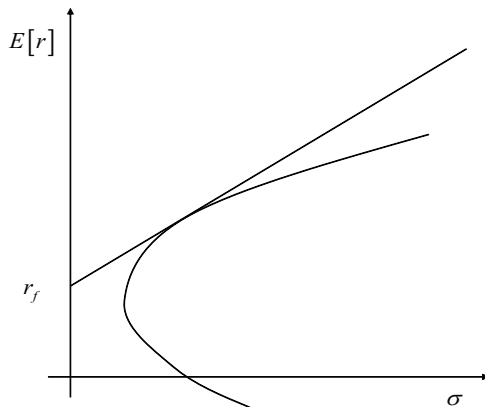


Figure: Efficient Expected Value-Standard Deviation Frontier with a Risk-Free Asset

Mechanics

- Setting up the portfolio problem

$$\begin{aligned} \min_w & \frac{1}{2} w' \Sigma w \\ \text{s.t. } & (\bar{z} - R1)' w = \mu - R \end{aligned} \quad (1)$$

- Forming the Lagrangian

$$\begin{aligned} L &= \frac{1}{2} w' \Sigma w - \gamma (\mu - R - (\bar{z} - R1)' w) \\ \nabla_w L &= \Sigma w - \gamma (\bar{z} - R1) = 0 \\ w^* &= \gamma \Sigma^{-1} (\bar{z} - R1) \end{aligned} \quad (2)$$

- With $w_0^* = 1 - 1' w^*$ defined as the amount of wealth invested in the risk-free asset if $w_0^* > 0$ or borrowed if $w_0^* < 0$.

- Substituting this result into the constraint yields

$$\begin{aligned} (\bar{z} - R1)' [\gamma \Sigma^{-1} (\bar{z} - R1)] &= \mu - R \\ \gamma [\bar{z}' \Sigma^{-1} \bar{z} - R \bar{z}' \Sigma^{-1} 1 - R1' \Sigma^{-1} \bar{z} - R^2 1' \Sigma^{-1} 1] &= \mu - R \end{aligned} \quad (3)$$

- From Lecture 21 we have

$$\begin{aligned} A &= 1' \Sigma^{-1} 1 \\ B &= 1' \Sigma^{-1} \bar{z} \\ C &= \bar{z}' \Sigma^{-1} \bar{z} \end{aligned} \quad (4)$$

Yielding the optimal solution

$$\begin{aligned} \gamma [C - 2RB + R^2 A] &= \mu - R \\ \gamma &= \frac{\mu - R}{C - 2RB + R^2 A} \end{aligned} \quad (5)$$

- The optimal portfolio is then defined as

$$w^* = \frac{\mu - R}{C - 2RB + R^2A} \Sigma^{-1} (\bar{z} - R1) \quad (6)$$

- The variance of the optimal portfolio then becomes

$$\begin{aligned}\sigma^2 &= w' \Sigma w \\&= \frac{(\mu - R)^2}{[C - 2RB - R^2A]^2} [\Sigma^{-1}(\bar{z} - R1)]' \Sigma [\Sigma^{-1}(\bar{z} - R1)] \\&= \frac{(\mu - R)^2}{[C - 2RB - R^2A]^2} (\bar{z} - R1)' \Sigma^{-1} \Sigma \Sigma^{-1} (\bar{z} - R1) \\&= \frac{(\mu - R)^2}{[C - 2RB - R^2A]^2} [\bar{z}' \Sigma^{-1} \bar{z} - R1' \Sigma^{-1} \bar{z} - R \bar{z}' \Sigma^{-1} 1 + R^2 1' \Sigma^{-1} 1] \\&= \frac{(\mu - R)^2}{[C - 2RB - R^2A]^2} [C - 2RB - R^2A] \\&= \frac{(\mu - R)^2}{[C - 2RB - R^2A]}\end{aligned}$$

(7)

Capital Market Line

- Note

$$\sigma = (\mu - R) \frac{1}{\sqrt{C - 2RB + R^2A}} \quad (8)$$

$\therefore \mu = R + (C - 2RB + R^2A)^{\frac{1}{2}} \Rightarrow$ Capital Market Line

- Or the tradeoff between expected return and the standard deviation is linear.
- The optimal portfolios with a risk-free asset are presented in Table 1.
- Table 2 depicts the relative portfolio share (i.e., dividing holding by the sum of the holdings).

Table: Shares in Portfolio with Risk-free Asset

Mean	Std Dev	ADM	3M	HP	John Deere
0.020	0.006	-0.178	0.062	0.084	0.084
0.030	0.029	-0.888	0.308	0.418	0.420
0.040	0.051	-1.598	0.555	0.752	0.756
0.050	0.074	-2.308	0.801	1.087	1.093
0.060	0.097	-3.018	1.048	1.421	1.429
0.070	0.120	-3.728	1.295	1.755	1.765
0.080	0.143	-4.438	1.541	2.089	2.101

Table: Relative Share of Risk-free Portfolio

Risk-free	ADM	3M	HP	John Deere
0.948	-3.428	1.191	1.614	1.623
0.741	-3.428	1.191	1.614	1.623
0.534	-3.428	1.191	1.614	1.623
0.327	-3.428	1.191	1.614	1.623
0.12	-3.428	1.191	1.614	1.623
-0.087	-3.428	1.191	1.614	1.623
-0.294	-3.428	1.191	1.614	1.623

Figure: Efficient Portfolio with Risk-free Asset

