

Derivation of the Expected Value-Variance Frontier without a Risk-free Asset : Lecture XXI

Charles B. Moss

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Expected Value-Variance Versus Direct Utility Maximization

- Due to various financial economic models such as the Capital Asset Pricing Model that we will discuss in our discussion of market models, the finance literature relies on the use of mean-variance decision rules rather than direct utility maximization.
- There is a practical aspect for stock-brokers who may want to give clients alternatives between efficient portfolios rather than attempting to directly elicit each individual's utility function.
- Kroll, Levy, and Markowitz examines the acceptability of the Mean-Variance procedure whether the expected utility maximizing choice is contained in the Mean-Variance efficient set.
- We assume that the decision maker is faced with allocating a stock portfolio between various investments.

Comparing Specifications Numerically

- Two approaches for making this problem are to choose between the set of investments to maximize expected utility

$$\begin{aligned} \max_x & E[U(x)] \\ \text{s.t.} & \sum_{i=1}^n x_i = 1 \\ & x_i \geq 0 \end{aligned} \quad (1)$$

- The second alternative is to map out the efficient Mean-Variance space by solving

$$\begin{aligned} \max_x & c'x \\ \text{s.t.} & x' \Omega x \leq t \\ & x_i \geq 0 \end{aligned} \quad (2)$$

- A better formulation of the problem is

$$\begin{aligned} \max_x \quad & c'x - \frac{\rho}{2}x'\Omega x \\ \text{s.t.} \quad & x_i \geq 0 \end{aligned} \tag{3}$$

where ρ is the Arrow-Pratt absolute risk aversion coefficient.

- Table 1 presents the optimal shares of the portfolio based on direct expected utility maximization and expected value variance using the formulation in Equation 3 with the equivalent risk aversion coefficients.

Table: Comparison of Expected Utility Maximization and Expected Value-Variance

Utility Function	California	Carpenter	Chrysler	Conelco	Texas Gulf	Average Return	Standard Deviation
<i>Direct Expected Utility Portfolios</i>							
$-\exp(-x)$	44.3	34.7	0.2	5.5	15.3	22.4	27.3
$x^{0.1}$	33.2	36.0		13.6	17.2	23.3	32.3
$x^{0.5}$		42.2		34.4	23.4	25.9	49.4
$\ln(x)$	37.9	34.8		11.1	16.2	23.1	29.4
<i>Expected Value-Variance Portfolios</i>							
$-\exp(-x)$	39.4	38.6		5.0	17.0	22.5	27.0
$x^{0.1}$	28.5	43.4		8.6	8.6	23.1	30.0
$x^{0.5}$		41.8		32.1	26.1	25.7	47.3
$\ln(x)$	32.9	41.8		7.4	18.7	22.9	28.9

Mathematical derivation without a Risk-free asset

- General formulation

$$\begin{aligned}\bar{z} &= w' \bar{z} + w_0 R = \sum_{i=0}^N w_i \bar{z}_i \\ \sigma^2 &= w' \Sigma w = \sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_{ij}\end{aligned}\tag{4}$$

- Mean-variance frontier without a risk-free asset

$$\begin{aligned}\min \quad & \frac{1}{2} w' \Sigma w \\ \text{s.t.} \quad & 1' w = 1 \\ & z' w = \mu\end{aligned}\tag{5}$$

- Lagrange formulation

$$L \equiv \frac{w' \Sigma w}{2} + \lambda (1 - 1' w) + \gamma (\mu - \bar{z}' w)$$

$$\nabla_w L = \nabla_w \left(\frac{w' \Sigma w}{2} \right) - \lambda \nabla_w (1' w) - \gamma (\bar{z}' w) \quad (6)$$

- Working through the variance

$$w' \Sigma w = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$= \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} w_1 \sigma_{11} + w_2 \sigma_{12} + w_3 \sigma_{13} \\ w_1 \sigma_{12} + w_2 \sigma_{22} + w_3 \sigma_{23} \\ w_1 \sigma_{13} + w_2 \sigma_{23} + w_3 \sigma_{33} \end{bmatrix} \quad (7)$$

$$= w_1 (w_1 \sigma_{11} + w_2 \sigma_{12} + w_3 \sigma_{13}) + w_2 (w_1 \sigma_{12} + w_2 \sigma_{22} + w_3 \sigma_{23}) + w_3 (w_1 \sigma_{13} + w_2 \sigma_{23} + w_3 \sigma_{33})$$

Gradients of the variance matrix

$$\begin{aligned}\nabla_w w' \Sigma w &= \begin{bmatrix} \frac{\partial w' \Sigma w}{\partial w_1} \\ \frac{\partial w' \Sigma w}{\partial w_2} \\ \frac{\partial w' \Sigma w}{\partial w_3} \end{bmatrix} \\ &= \begin{bmatrix} (w_1 \sigma_{11} + w_2 \sigma_{12} + w_3 \sigma_{13}) + w_1 \sigma_{11} + w_2 \sigma_{12} + w_3 \sigma_{13} \\ (w_1 \sigma_{12} + w_2 \sigma_{22} + w_3 \sigma_{23}) + w_1 \sigma_{12} + w_2 \sigma_{22} + w_3 \sigma_{23} \\ (w_1 \sigma_{13} + w_2 \sigma_{23} + w_3 \sigma_{33}) + w_1 \sigma_{13} + w_2 \sigma_{23} + w_3 \sigma_{33} \end{bmatrix} \\ &= \begin{bmatrix} 2(w_1 \sigma_{11} + w_2 \sigma_{12} + w_3 \sigma_{13}) \\ 2(w_1 \sigma_{12} + w_2 \sigma_{22} + w_3 \sigma_{23}) \\ 2(w_1 \sigma_{13} + w_2 \sigma_{23} + w_3 \sigma_{33}) \end{bmatrix}\end{aligned}$$

(8)

A Little Matrix Calculus

$$\nabla_w w' \Sigma w = \Sigma w + (w' \Sigma)'$$

$$\Sigma w = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad (9)$$

$$= \begin{bmatrix} w_1 \sigma_{11} + w_2 \sigma_{12} + w_3 \sigma_{13} \\ w_1 \sigma_{12} + w_2 \sigma_{22} + w_3 \sigma_{23} \\ w_1 \sigma_{13} + w_2 \sigma_{23} + w_3 \sigma_{33} \end{bmatrix}$$

$$w' \Sigma = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

$$= \begin{bmatrix} w_1\sigma_{11} + w_2\sigma_{12} + w_3\sigma_{13} & w_1\sigma_{12} + w_2\sigma_{22} + w_3\sigma_{23} & w_1\sigma_{13} + w_2\sigma_{23} + w_3\sigma_{33} \end{bmatrix}$$

$$(w' \Sigma)' = \begin{bmatrix} w_1\sigma_{11} + w_2\sigma_{12} + w_3\sigma_{13} \\ w_1\sigma_{12} + w_2\sigma_{22} + w_3\sigma_{23} \\ w_1\sigma_{13} + w_2\sigma_{23} + w_3\sigma_{33} \end{bmatrix}$$

(10)

$$\begin{aligned}\nabla_w w' \Sigma w &= \Sigma w + (w' \Sigma)' \\&= \begin{bmatrix} w_1 \sigma_{11} + w_2 \sigma_{12} + w_3 \sigma_{13} \\ w_1 \sigma_{12} + w_2 \sigma_{22} + w_3 \sigma_{23} \\ w_1 \sigma_{13} + w_2 \sigma_{23} + w_3 \sigma_{33} \end{bmatrix} + \begin{bmatrix} w_1 \sigma_{11} + w_2 \sigma_{12} + w_3 \sigma_{13} \\ w_1 \sigma_{12} + w_2 \sigma_{22} + w_3 \sigma_{23} \\ w_1 \sigma_{13} + w_2 \sigma_{23} + w_3 \sigma_{33} \end{bmatrix} \\&= \begin{bmatrix} 2(w_1 \sigma_{11} + w_2 \sigma_{12} + w_3 \sigma_{13}) \\ 2(w_1 \sigma_{12} + w_2 \sigma_{22} + w_3 \sigma_{23}) \\ 2(w_1 \sigma_{13} + w_2 \sigma_{23} + w_3 \sigma_{33}) \end{bmatrix} \tag{11}\end{aligned}$$

- Looking at the constraints

$$1'w = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = w_1 + w_2 + w_3$$

$$\alpha'w = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 \quad (12)$$

$$\nabla_w \alpha'w = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \Rightarrow \nabla_w 1'w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Returning to the Lagrange formulation

$$\begin{aligned}\nabla_w L &= \Sigma w - \lambda \mathbf{1} - \gamma \bar{z} = 0 \\ \Sigma w &= \lambda \mathbf{1} + \gamma \bar{z} \\ w^* &= \lambda \Sigma^{-1} + \gamma \Sigma^{-1} \bar{z}\end{aligned}\tag{13}$$

$$\Sigma^{-1} = [\sigma^{ij}]$$

$$\begin{aligned}w^* &= \lambda \begin{bmatrix} \sigma^{11} & \sigma^{12} & \sigma^{13} \\ \sigma^{12} & \sigma^{22} & \sigma^{23} \\ \sigma^{13} & \sigma^{23} & \sigma^{33} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} \sigma^{11} & \sigma^{12} & \sigma^{13} \\ \sigma^{12} & \sigma^{22} & \sigma^{23} \\ \sigma^{13} & \sigma^{23} & \sigma^{33} \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \end{bmatrix} \\ &= \begin{bmatrix} \lambda (\sigma^{11} + \sigma^{12} + \sigma^{13}) + \gamma (\sigma^{11} \bar{z}_1 + \sigma^{12} \bar{z}_2 + \sigma^{13} \bar{z}_3) \\ \lambda (\sigma^{12} + \sigma^{22} + \sigma^{23}) + \gamma (\sigma^{12} \bar{z}_1 + \sigma^{22} \bar{z}_2 + \sigma^{23} \bar{z}_3) \\ \lambda (\sigma^{13} + \sigma^{23} + \sigma^{33}) + \gamma (\sigma^{13} \bar{z}_1 + \sigma^{23} \bar{z}_2 + \sigma^{33} \bar{z}_3) \end{bmatrix}\end{aligned}\tag{14}$$

- Solving the constraints

$$\begin{aligned}\bar{z}'w^* &= \bar{z}' [\lambda \Sigma^{-1} \mathbf{1} + \gamma \Sigma^{-1} \bar{z}] = \mu \\ \lambda \bar{z}' \Sigma^{-1} \mathbf{1} + \gamma \bar{z}' \Sigma^{-1} \bar{z} &= \mu\end{aligned}\tag{15}$$

- Letting

$$B = \bar{z}' \Sigma^{-1} \mathbf{1} = \mathbf{1}' \Sigma^{-1} \bar{z} <=> 0$$

$$C = \bar{z}' \Sigma^{-1} \bar{z} > 0$$

$$\lambda B + \gamma C = \mu\tag{16}$$

$$\gamma^* = \frac{\mu - \lambda B}{C}$$

$$w^* = \left[\lambda \Sigma^{-1} \mathbf{1} + \left(\frac{\mu - \lambda B}{C} \right) \Sigma^{-1} \bar{z} \right]$$

- Solving for λ^*

$$1' \left[\lambda \Sigma^{-1} 1 + \left(\frac{\mu - \lambda B}{C} \right) \Sigma^{-1} \bar{z} \right] = 1$$

$$\lambda 1' \Sigma^{-1} 1 + \left(\frac{\mu - \lambda B}{C} \right) 1' \Sigma^{-1} \bar{z} = 1$$

$$1' \Sigma^{-1} 1 \equiv A > 0$$

$$\lambda A + \left(\frac{\mu - \lambda B}{C} \right) B = 1 \quad (17)$$

$$\lambda AC + \mu B - \lambda B^2 = C$$

$$\lambda (AC - B^2) = C - \mu B$$

$$\lambda^* = \frac{C - \mu B}{AC - B^2}$$

- Solving for γ^*

$$\begin{aligned}\gamma^* &= \frac{\mu - \left(\frac{C - \mu B}{AC - B^2} \right) B}{C} \\&= \frac{(AC - B^2) \mu - (C - \mu B) B}{C (AC - B^2)} \\&= \frac{\mu AC - B^2 \mu - CB + \mu B^2}{C (AC - B^2)} \\&= \frac{\mu A - B}{AC - B^2}\end{aligned}\tag{18}$$

Stock portfolio example

- From the stock example from the last lecture

$$\Sigma = \begin{bmatrix} 0.00316 & 0.00129 & 0.00355 & 0.00265 \\ 0.00129 & 0.00226 & 0.00129 & 0.00171 \\ 0.00265 & 0.00129 & 0.00670 & 0.00155 \\ 0.00265 & 0.00171 & 0.00155 & 0.03369 \end{bmatrix} \quad (19)$$

$$\bar{z} = \begin{bmatrix} 0.02041 \\ 0.02988 \\ 0.02818 \\ 0.03369 \end{bmatrix}$$

- Fixing $\mu = 0.05$

$$A = 523.825$$

$$B = 13.1262$$

$$C = 0.49018$$

$$\lambda = \frac{C - \mu B}{AC - B^2} = -0.001967 \quad (20)$$

$$\gamma = \frac{\mu A - B}{AC - B^2} = 0.15467$$

- Optimal Portfolio

$$w^* = \begin{bmatrix} -1.90806 \\ 0.98955 \\ 0.954418 \\ 0.964088 \end{bmatrix} \Rightarrow \sigma^2 = 0.005767 \quad (21)$$

- Fixing $\mu = 0.04$, note A , B , and C remain unchanged

$$\lambda = -0.0004128$$

$$\gamma = 0.09266$$

$$w^* = \begin{bmatrix} -0.9450 \\ 0.8621 \\ 0.5364 \\ 0.5465 \end{bmatrix} \Rightarrow \sigma^2 = 0.003293 \quad (22)$$