

Portfolio and Risk Analysis: Lecture XX

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An Overview of Portfolio

- The benefit to holding a portfolio of assets appears to follow the old adage: **Don't put all of your eggs in one basket.**
 - However, what are the mathematics?
 - Data on the returns for Archer-Daniels-Midland, 3M, Hewlett Packard, and John Deere are taken from the WRDS (Wharton Business School) database is presented in Table 1.

Table: Portfolio Example

Date	ADM	3M	HP	Deere	Portfolio
1/31/2003	-0.0282	0.0101	0.0029	-0.0796	-0.0030
2/28/2003	-0.0905	0.0119	-0.0896	-0.0194	-0.0061
3/31/2003	-0.0092	0.0372	-0.0139	-0.0459	0.0195
4/30/2003	0.0259	-0.0307	0.0482	0.1215	-0.0006
5/30/2003	0.0857	0.0086	0.1963	-0.0082	0.0301
6/30/2003	0.0752	0.0198	0.0964	0.0515	0.0340
7/31/2003	0.0210	0.0870	-0.0061	0.1112	0.0779
8/29/2003	0.0601	0.0209	-0.0586	0.1128	0.0241
9/30/2003	-0.0548	-0.0304	-0.0246	-0.0527	-0.0330
10/31/2003	0.0946	0.1419	0.1524	0.1371	0.1417
11/28/2003	0.0000	0.0063	-0.0255	0.0101	0.0029
12/31/2003	0.0651	0.0758	0.0603	0.0660	0.0725
Average	0.0204	0.0299	0.0282	0.0337	0.0300
Variance	0.0035	0.0025	0.0073	0.0058	0.0023
Portfolio	0.0178	0.7339	0.1188	0.1295	1.0000

Optimal Portfolio

- Define the "optimal portfolio" as

$$\begin{aligned} \min_z V \left(\sum_{i=1}^{12} z_i r_i \right) \\ \text{s.t. } \sum_{i=1}^{12} z_i \bar{r}_i = 0.03 \\ \sum_{i=1}^{12} z_i = 1.00 \end{aligned} \quad (1)$$

where

$$\begin{aligned} V \left(\sum_{i=1}^{12} z_i r_i \right) &= \sum_{t=1}^{12} \left(\sum_{i=1}^4 z_i r_{it} - \sum_{i=1}^4 z_i \bar{r}_i \right)^2 \\ \bar{r}_i &= \frac{1}{12} \sum_{i=1}^{12} r_{it} \end{aligned} \quad (2)$$

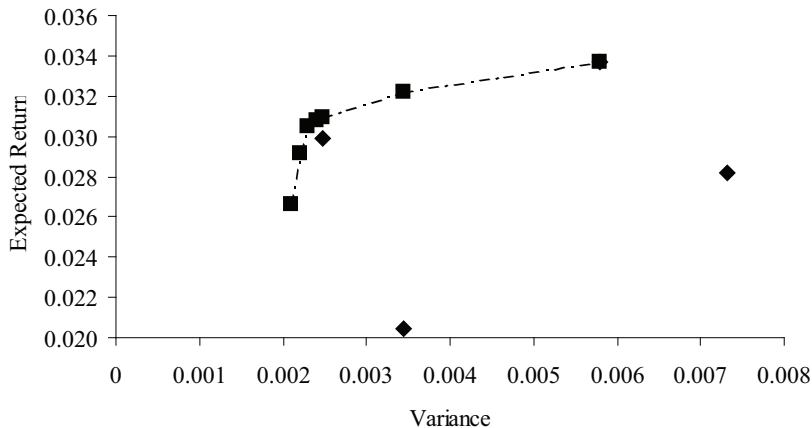
- From an expected value-variance sense one asset dominates another asset if it produces the same expected value for a lower variance.
 - This definition of dominance follows from our discussion of Meyers and the basic concept of expected utility under normality.
 - In this case, 3M dominates both Archer-Daniels-Midland and Hewlett Packard.
 - We cannot directly compare 3M and John Deere because John Deere implies a higher risk than 3M, but also implies a higher rate of return.
 - Note that the portfolio dominates not only Archer-Daniels-Midland and Hewlett Packard, but the portfolio also dominates 3M.
 - However, the portfolio does not dominate John Deere.

The Expected Value-Variance Frontier

- Resolving the portfolio problem as

$$\begin{aligned} \max_z \quad & \sum_{i=1}^{12} z_i \bar{r}_i \\ \text{s.t.} \quad & V \left(\sum_{i=1}^{12} z_i r_i \right) \leq \sigma^2 \\ & \sum_{i=1}^{12} z_i = 1.00 \end{aligned} \tag{3}$$

- The Expected Value-Variance frontier from this formulation is presented in Figure 1



◆ Observed Returns --■-- Portfolio

Assuming Normality

- Reformulating the problem using the normal distribution function

$$\begin{aligned} \min_x \sigma_P^2(x) &= x' \Sigma x \\ \text{s.t. } x' \bar{r} &\geq \mu \\ x &\geq 0 \end{aligned} \quad (4)$$

- The mean vector can be depicted as

$$\begin{aligned} \bar{r} &= \frac{1}{n} 1' r = \frac{1}{n} \left(\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}' \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nm} \end{bmatrix} \right)' \\ &= \frac{1}{n} \begin{bmatrix} \sum_{i=1}^n r_{i1} \\ \vdots \\ \sum_{i=1}^n r_{im} \end{bmatrix} \end{aligned} \quad (5)$$

- Continued
 - The variance matrix is then defined as

$$\Sigma = \frac{1}{n-1} r' r - \bar{r} \bar{r}'$$

$$\bar{r} = \begin{bmatrix} 0.02041 \\ 0.02988 \\ 0.02818 \\ 0.03369 \end{bmatrix} \quad (6)$$

$$\Sigma = \begin{bmatrix} 0.00316 & 0.00129 & 0.00355 & 0.00265 \\ 0.00129 & 0.00226 & 0.00129 & 0.00171 \\ 0.00355 & 0.00129 & 0.00670 & 0.00155 \\ 0.00265 & 0.00171 & 0.00155 & 0.00532 \end{bmatrix}$$

The frontier given this mean vector and variance matrix is presented in Figure 2.

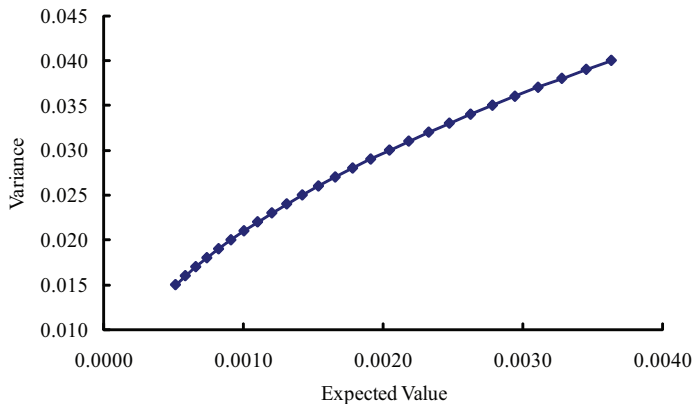


Figure: Continuous Expected Value-Variance Frontier

Portfolio Selection by Utility Maximization

- Returning to the expected utility model, the portfolio choice model can be cast as

$$\begin{aligned} \max_{c_1, h} & E[v(c_1)] \\ \text{s.t.} & ph = w_0 \\ c_1 = & w_1 + \sum_{j=1}^J x_j h_j \end{aligned} \tag{7}$$

- Next, we replace the portfolio with a single asset

$$r = \frac{\sum_{j=1}^J x_j h_j}{\sum_{j=1}^J p_j h_j} \tag{8}$$

- In this case, the original problem can be simplified to

$$\max_a E[v(w\bar{r} + (r - \bar{r})a)] \quad (9)$$

The optimality is defined by the tangency between the utility function and the expected value-variance frontier as depicted in Figure 3.

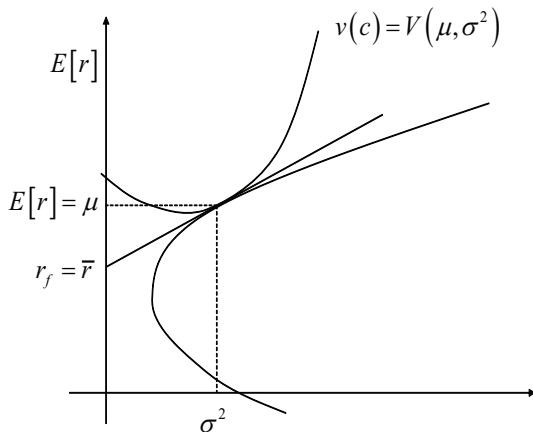


Figure: Utility Tangency with the Expected Value-Variance Frontier

- The relationship between the point that maximizes expected utility and the expected value-variance frontier has been discussed (implicitly) in the Freund formulation and Meyer's location-scale formulation. In each case we defined the expected utility of the gamble as

$$v(y) = -(\alpha - y)^2 \quad (10)$$

- Equations 9 and 10 imply the the first-order condition with respect to a (the level of risky asset held) of

$$\begin{aligned} E[(\alpha - w\bar{r} - a\{r - \bar{r}\})(r - \bar{r})] &= 0 \\ E[(\alpha - w\bar{r})(r - \bar{r}) - a(r - \bar{r})^2] &= 0 \\ (\alpha - w\bar{r})(\mu - \bar{r}) - aE[(r - \mu + \mu - \bar{r})^2] &= 0 \\ (\alpha - w\bar{r})(\mu - \bar{r}) - aE[(\{r - \mu\} + \{\mu - \bar{r}\})^2] &= 0 \\ (\alpha - w\bar{r})(\mu - \bar{r}) - aE[(r - \mu)^2 + 2(r - \mu)(\mu - \bar{r}) + (\mu - \bar{r})^2] &= 0 \end{aligned} \quad (11)$$

- Continued
 - Given

$$\begin{aligned}E[(r - \mu)^2] &= \sigma^2 \\E[(r - \mu)(\mu - \bar{r})] &= 0 \\E[(\mu - \bar{r})^2] &= (\mu - \bar{r})^2\end{aligned}\tag{12}$$

The optimal share of risk assets can be derived as

$$a^* = \frac{(\alpha - w\bar{r})(\mu - \bar{r})}{\sigma^2 + (\mu - \bar{r})}\tag{13}$$

- **Theorem 5.1** *p-109* If an agent is strictly risk averse, then the optimal investment in the risky security is strictly positive, zero, or strictly negative if and only if the risk premium on the risky security is strictly positive, zero, or strictly negative.