

Utility Functions, Risk Aversion Coefficients and Transformations: Lecture XVIII

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1 Arrow-Pratt Coefficients for particular functions

- Quadratic Utility Function
- Power Utility Function
- Negative Exponential Utility
- Hyperbolic Absolute Risk Aversion

2 Interpretations and Transformations

Arrow-Pratt Coefficients for particular functions

- Quadratic Utility Function: To specify the appropriate shape of the utility function, the quadratic function becomes

$$\begin{aligned}U(w) &= aw - bw^2 \\U'(w) &= a - 2bw \\U''(w) &= -2b\end{aligned}\tag{1}$$

Arrow-Pratt absolute risk aversion coefficient

$$\begin{aligned}R_A(w) &= -\frac{-2b}{a - 2bw} \Rightarrow R_A(w) = \frac{2b}{a - 2bw} \\ \frac{dR_A(w)}{dw} &= 2b \frac{2b}{(a - 2bw)^2} > 0 \Leftarrow \frac{d}{dx} \left[\frac{1}{f(x)} \right] = -\frac{f'(x)}{(f(x))^2}\end{aligned}\tag{2}$$

- Arrow-Pratt relative risk aversion coefficient

$$R_R(w) = \left[\frac{2b}{a - 2bw} \right] w = \frac{2b}{\frac{a}{w} - 2b} \quad (3)$$
$$\frac{dR_R(w)}{dw} = 2b \frac{2b}{(a - 2bw)^2} + \frac{2b}{a - 2bw} > 0$$

- Power Utility Function

$$U(w) = \frac{w^{1-r}}{1-r}$$
$$U'(w) = w^{-r} \tag{4}$$

$$U''(w) = -rw^{-r-1}$$

Arrow-Pratt absolute risk aversion coefficient

$$R_A(w) = - \left[\frac{-rw^{-r-1}}{w^{-r}} \right] = \frac{r}{w}$$
$$\frac{dR_A(w)}{dw} = -\frac{r}{w^2} < 0 \tag{5}$$

- Arrow-Pratt relative risk aversion coefficient

$$R_R(w) = - \left[\frac{rw^{-r-1}w}{w^{-r}} \right] = r \quad (6)$$
$$\frac{dR_R(w)}{dw} = 0$$

Constant relative risk aversion.

- Negative Exponential Utility Function

$$\begin{aligned}U(w) &= -\exp(-\rho w) \\U'(w) &= \rho \exp(-\rho w) \\U''(w) &= -\rho^2 \exp(w)\end{aligned}\tag{7}$$

Arrow-Pratt absolute risk aversion coefficient

$$R_A(w) = -\left[\frac{-\rho \exp(-\rho w)}{\rho \exp(-\rho w)}\right] = \rho\tag{8}$$

- Constant absolute risk aversion. Arrow-Pratt relative risk aversion coefficient

$$R_R(w) = - \left[\frac{-\rho^2 \exp(-\rho w)}{\rho \exp(-\rho w)} w \right] = \rho w \quad (9)$$
$$\frac{dR_R(w)}{dw} = \rho > 0$$

- HARA-Hyperbolic Absolute Risk Aversion

$$U(w) = \frac{1-\gamma}{\gamma} \left(\frac{aw}{1-\gamma} + b \right)^{\gamma}, b > 0$$

$$\begin{aligned} U'(w) &= (1-\gamma) \left(\frac{aw}{1-\gamma} + b \right)^{\gamma-1} \left(\frac{a}{1-\gamma} \right) \\ &= a \left(\frac{aw}{1-\gamma} + b \right)^{\gamma-1} \end{aligned} \tag{10}$$

$$\begin{aligned} U''(w) &= a(\gamma-1) \left(\frac{aw}{1-\gamma} + b \right)^{\gamma-2} \left(\frac{a}{1-\gamma} \right) \\ &= a^2 \left(\frac{aw}{1-\gamma} + b \right)^{\gamma-2} \end{aligned}$$

- Arrow-Pratt absolute risk aversion coefficient

$$\begin{aligned}
 R_A(w) &= - \left[\frac{-a^2 \left(\frac{aw}{1-\gamma} + b \right)^{\gamma-2}}{a \left(\frac{aw}{1-\gamma} + b \right)^{\gamma-1}} \right] \\
 &= \frac{a}{\left(\frac{aw}{1-\gamma} + b \right)} \\
 &= \frac{a}{\frac{aw}{1-\gamma} + \frac{b(1-\gamma)}{1-\gamma}} \\
 &= \frac{a(1-\gamma)}{aw + b(1-\gamma)}
 \end{aligned} \tag{11}$$

- As $\gamma \rightarrow 1$ the HARA becomes risk neutral.
- $\gamma = 2$ is a quadratic.
- $\gamma \rightarrow -\infty$ and $b = 1$ is the negative exponential.
- $b = -$ and $\gamma < 1$ is the power utility function.

Interpretations and Transformations

- To this point, we have discussed technical manifestations of risk aversion such as where the risk aversion coefficient comes from and how the utility of income is derived. However, I want to start turning to the question: How do we apply the concept of risk aversion?
- Several procedures exist for integrating risk into the decision making process such as direct application of expected utility, mathematical programming using the expected value-variance approximation, or the use of stochastic dominance. All of these approaches, however, require some notion of the relative size of risk aversion.

- Risk aversion directly uses a risk aversion coefficient to parameterize the negative exponential or power utility functions.
- Mathematical programming uses the concept of the tradeoff between variance and expected income.
- Stochastic dominance uses measures of risk aversion to bound the utility function.

- The current study gives some guidance on using previously published risk aversion coefficients. Specifically, the article looks at the effect of location and scale on the risk aversion coefficient.
 - As a starting place, we develop an interpretation of the Pratt-Arrow coefficient in terms of marginal utility.

$$\begin{aligned} R_R(x) &= -\frac{U''(x)}{U'(x)} \\ &= -\frac{\left(\frac{dU'(x)}{dx}\right)}{U'(x)} \\ &= -\frac{d}{dx} \ln(U'(x)) = -\frac{\left(\frac{dU'(x)}{U'(x)}\right)}{dx} \end{aligned} \tag{12}$$

- This algebraic manipulation develops the absolute risk aversion coefficient as the percent change in marginal utility at any level of income.
- Therefore, r is associated with a unit of change in outcome space. If the risk aversion coefficient was elicited in outcomes of dollars, then the risk aversion coefficient is $.0001/\$$.

- **Theorem 1:** Let $R_R(x) = U''(x) / U'(x)$. Define a transformation of scale on x such that $w = x/c$, where c is a constant. Then $R_R(w) = cR_R(x)$.
- The proof lies in the change of variables. Given

$$\begin{aligned}
 U(x) &= U(wc) \\
 \frac{dU}{dw} &= \frac{dU}{dx} \frac{dx}{dw} = cU'(x) \\
 \frac{d^2U}{dw^2} &= \frac{d}{dw} \left(\frac{dU}{dw} \right) = \frac{d}{dw} (cU'(x)) = c^2 U''(x)
 \end{aligned} \tag{13}$$

- In other words, if the scale of the outcome changes by c , the scale of the risk aversion coefficient must be changed by the same amount.
- **Theorem 2:** If $v = x + c$, where c is a constant, then $R_R(v) = R_R(x)$. Therefore, the magnitude of the risk aversion coefficient is unaffected by the use of incremental rather absolute returns.

- Example: Suppose that a study of U.S. farmers gives a risk aversion coefficient of $r = .0001/\$$ (U.S.) Application to the Australian farmers whose dollar is worth .667 of the U.S. dollar is $r = .0000667/\$$ Australian.