

Risk Aversion in the Large and Small: Lecture XVII

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Basics of Risk Aversion

- Back to Friedman and Savage:
 - An economic agent with a von Neumann-Morgenstern utility function $v : R \rightarrow R$ is weakly risk averse if and only if

$$E[v(c)] \leq v(E[c]) \quad (1)$$

for every consumption plan c .

- An economic agent is risk neutral if

$$E[v(c)] = v(E[c]) \quad (2)$$

- Similarly, an economic agent is strictly risk averse if

$$E[v(c)] < v(E[c]) \quad (3)$$

Risk Aversion and Concavity

- Risk Aversion and Concavity
 - **Theorem 4.1** *p-87*
 - The agent is risk averse if and only if his or her von Neumann-Morgenstern utility function v is concave.
 - An agent is risk neutral if and only if his or her von Neumann-Morgenstern utility function v is linear.
 - An agent is strictly risk averse if and only if his or her von Neumann-Morgenstern utility function v is strictly concave.

Proof

- To justify this theorem we start with the basic concept of concavity. If the utility function is concave

$$u(\alpha x_1 + (1 - \alpha) x_2) \geq \alpha u(x_1) + (1 - \alpha) u(x_2) \quad (4)$$

- This relationship is depicted graphically in Figure 1
- Interpreting α as a probability completes the linkage between concavity and risk aversion.

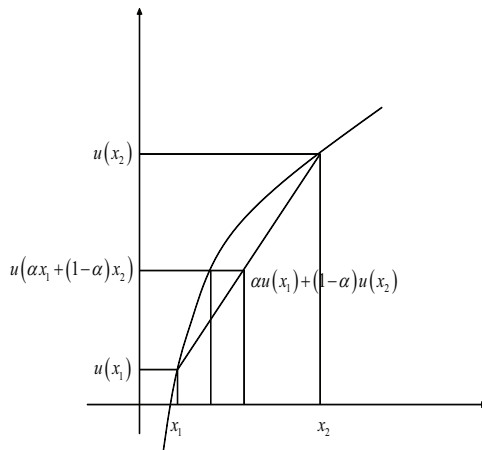


Figure: Graphical Depiction of Risk Aversion

Arrow-Pratt

- The first step is to establish the existence of a risk-premium. The risk premium $\pi(x, z)$ is the certain payment that leaves the decision maker indifferent between the gamble and the risk-free outcome.
- Mathematically, the risk premium is defined as

$$u(x + E(z) - \pi(x, z)) = E\{u(x + z)\} \quad (5)$$

- The first step is to normalize $E[z]$. We note that

$$\pi(x, z) = \pi(x + \mu, z - \mu) \quad (6)$$

Thus, the expected value of the risky gamble is just added to the base endowment of x .

- Note that the investor is indifferent between receiving the risky return z and the sure amount

$$\pi_a(x, z) = E[z] - \pi(x, z) \quad (7)$$

$\pi_a(x, z)$ is called the certainty equivalent of the gamble. The certainty equivalent is defined by

$$u(x + \pi_a(x, z)) = E\{u(x + z)\} \quad (8)$$

Local Risk Aversion

- Given this formulation, it is possible to define the local risk aversion for an investor.
- Going back to the original equation for the risk premium, ignoring for the moment $E(z)$, the utility function can be written as

$$u(x - \pi) = u(x) - \pi u'(x) + O(\pi^2) \quad (9)$$

The right-hand side of this expression is simply the first order Taylor expansion.

- Taking the second order expansion of the expected side of the expression yields

$$E\{U(x + z)\} = E\left\{u(x) + zu'(x) + \frac{1}{2}z^2u''(x) + O(z^3)\right\}$$

- Setting these expressions equal to one another, as depicted in the equation yields

$$u(x) - \pi u'(x) = u(x) + \frac{1}{2} \sigma_z^2 u''(x) \quad (11)$$
$$\pi(x, z) = -\frac{1}{2} \sigma_z^2 \frac{u''(x)}{u'(x)}$$

- The last part of the expression defines local risk aversion

$$r(x) = -\frac{u''(x)}{u'(x)} = -\frac{d}{dx} \ln(u'(x)) \quad (12)$$

Relative and Absolute Risk Aversion

- From this formulation, it is apparent that primary measure of the relative risk aversion involves the concavity of the utility function. There are two dominant measures, the absolute Arrow-Pratt risk aversion coefficient

$$R_A(x) = -\frac{U''(x)}{U'(x)} \quad (13)$$

and the relative Arrow-Pratt risk aversion coefficient

$$R_R(x) = -\frac{xU''(x)}{U'(x)} \quad (14)$$

- Two utility functions demonstrate the alternative forms of risk aversion.
 - The first utility function is the negative exponential utility function that depicts constant relative risk aversion. Specifically,

$$U(x) = -\exp(-\rho x) \quad (15)$$

The absolute risk aversion coefficient becomes

$$R_A(x) = -\frac{-\rho^2 \exp(\rho x)}{\rho \exp(-\rho x)} = \rho \quad (16)$$

- Continued

- The utility function which possesses constant relative risk aversion is the power utility function depicted as

$$U(x) = \frac{x^r}{r} \quad (17)$$

The constant relative risk aversion coefficient then becomes

$$R_R(x) = -\frac{x(r-1)x^{r-2}}{x^{r-1}} = -r - 1 \quad (18)$$

- The difference between the two can be demonstrated by looking at the value of two representative gambles.
 - The first gamble is based on a wealth of \$10,000. There are two possible outcomes for this gamble. The first outcome yields 90% of the decision makers original wealth or \$9,000.
 - The second potential outcome yields 1.25% of the decision maker's original wealth or \$12,500. Next we posit a second gamble based on the same probabilities and rates of return, but we assume that the second individual is endowed with \$100,000.
 - Table 1 gives the expected value and certainty equivalence for each scenario under constant relative risk aversion.

Table: Certainty Equivalents

Initial Equity	Expected Value	$r = 0.25$	$r = 0.01$	$\theta = 0.00025$
\$10,000	\$11,625	\$11,543	\$6,574	\$11,300
\$100,000	\$116,250	\$115,432	\$65,743	\$95,543

- These scenarios show a proportionality between the constant absolute risk aversion scenarios while no such proportionality is demonstrated under the constant absolute risk aversion scenario.
- Another manifestation of the different implications of these functions is the absolute risk aversion coefficient of the power utility function. Specifically,

$$R_A(x) = \frac{(r-1)x^{r-2}}{x^{r-1}} = \frac{r-1}{x} \quad (19)$$